Uncertainty Quantification and Quasi-Monte Carlo — Exercise 1

Wintersemester 2022/23

Return your written solutions either in person or by email to vesa.kaarnioja@fu-berlin.de by Tuesday 25 October, 2022, 12:15

1. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and $\|\cdot\|$ is the induced norm. Prove that an inner product satisfies the so-called parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all } x, y \in X.$$

2. (a) Consider the space $C([a,b])$. Show that the sup-norm

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$$

is not determined by any inner product.

(b) Similarly consider the L^p norms

$$\|f\|_p = \bigg(\int_a^b |f(x)|^p \,\mathrm{d}x\bigg)^{1/p},$$

where $1 \leq p < \infty$. Prove that if $p \neq 2$, then this norm is never determined by an inner product.

Hint: Any norm induced by an inner product satisfies the parallelogram identity.

3. Let $(X_i, \|\cdot\|_i)$ be normed spaces, $i \in \{1,2,3\}$. Show that for the norm of a linear operator $A\colon X_1 \to X_2$, we have

$$\|A\| = \sup_{0\neq\|x\|_1\leq 1} \frac{\|Ax\|_2}{\|x\|_1} = \sup_{\|x\|_1=1} \frac{\|Ax\|_2}{\|x\|_1}.$$

Let also $B\colon X_2 \to X_3$ be linear. Prove that

$$\|BA\| \leq \|B\|\|A\|.$$

4. Let H be a real Hilbert space. Recall that the *orthogonal complement* of any subset $M \subset H$ is defined as

$$M^\perp := \{y \in H \mid \langle x, y\rangle = 0 \text{ for all } x \in M\}.$$

(a) Show that for any subset $M \subset H$, $M^\perp$ is a closed subspace of H and $M \subset (M^\perp)^\perp$.

(b) If M is a non-closed subspace of H, show that $(M^\perp)^\perp = \overline{M}$, where the bar denotes the closure of a set. What if M is a closed subspace of H?