Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Tuesday 24 January, 2023, 12:15

Continuing from last week's exercises, we consider an elliptic PDE problem. Let $D \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, be a bounded Lipschitz domain and let $f \in L^2(D)$ be fixed. For all $\boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$, let $u(\cdot,\boldsymbol{y}) \in H_0^1(D)$ be such that

$$\int_D a(\boldsymbol{x},\boldsymbol{y})\nabla u(\boldsymbol{x},\boldsymbol{y})\cdot\nabla v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_D f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{for all } v \in H_0^1(D),$$

with the diffusion coefficient

$$a(\boldsymbol{x},\boldsymbol{y}) = a_0(\boldsymbol{x}) + \sum_{j=1}^{\infty} y_j\psi_j(\boldsymbol{x}), \quad \boldsymbol{x}\in D,\ \boldsymbol{y} = (y_j)_{j\geq 1} \in [-1/2,1/2]^{\mathbb{N}},$$

where we assume the following:

(A1) There exist constants $a_{\min}, a_{\max} > 0$ such that $0 < a_{\min} \leq a(\boldsymbol{x},\boldsymbol{y}) \leq a_{\max} < \infty$ for all $\boldsymbol{x} \in D$ and $\boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$.

(A2) $a_0 \in L^\infty(D)$ and $\psi_j \in L^\infty(D)$ for all $j \geq 1$ such that $\sum_{j\geq 1}\|\psi_j\|_{L^\infty(D)} < \infty$.

(A3) For some $p \in (0,1)$, there holds $\sum_{j\geq 1}\|\psi_j\|_{L^\infty(D)}^p < \infty$.

Moreover, we define the dimensionally-truncated solution by setting $u_s(\cdot,(y_1,\ldots,y_s)) := u(\cdot,(y_1,\ldots,y_s,0,0,\ldots))$ for $y_j \in [-1/2,1/2]$, $1 \leq j \leq s$.

Added 21.1.: Tasks 1–2 make use of the regularity bound derived for $\partial^{\boldsymbol{\nu}}u(\cdot,\boldsymbol{y})^2$ last week. Unfortunately, this result is *incorrect*. You may complete tasks 1–2 this week by assuming that the stated regularity bound holds (formally) for $\partial^{\boldsymbol{\nu}}u(\cdot,\boldsymbol{y})^2$ OR you can solve tasks 1–2 using instead the function

$$v(\boldsymbol{y}) := \int_D u(\boldsymbol{x},\boldsymbol{y})^2\,\mathrm{d}\boldsymbol{x}, \quad \boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}, \tag{1}$$

which *does* satisfy the estimates

$$\begin{aligned}|\partial^{\boldsymbol{\nu}}v(\boldsymbol{y})| &\leq C(|\boldsymbol{\nu}|+1)!\boldsymbol{b}^{\boldsymbol{\nu}} \quad \text{for all } \boldsymbol{\nu}\in\mathscr{F} \text{ and } \boldsymbol{y}\in[-1/2,1/2]^{\mathbb{N}},\\ \|v(\cdot-\tfrac{\mathbf{1}}{\mathbf{2}})\|_{s,\boldsymbol{\gamma}}^2 &\leq C'\sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}}\frac{[(|\mathfrak{u}|+1)!]^2}{\gamma_{\mathfrak{u}}}\prod_{j\in\mathfrak{u}}b_j^2.\end{aligned} \tag{2}$$

for certain generic constants $C, C' > 0$. For further details, see `https://vesak90.userpage.fu-berlin.de/erratum.PNG`.
I apologize for the inconvenience.

Your tasks are outlined below.

1. Recall from the lectures that, for given dimension $s \geq 1$, number of QMC nodes $n \geq 2$, and sequence of positive weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subseteq\{1,\ldots,s\}}$, the generating vector $\boldsymbol{z} \in \mathbb{U}_n^s$ obtained using the CBC algorithm satisfies the error bound

$$\sqrt{\mathbb{E}_{\boldsymbol{\Delta}}|I_sF - Q_{n,s}^{\boldsymbol{\Delta}}F|^2} \leq \left(\frac{1}{\varphi(n)} \sum_{\varnothing\neq\mathfrak{u}\subseteq\{1,\ldots,s\}} \gamma_{\mathfrak{u}}^{\lambda}\left(\frac{2\zeta(2\lambda)}{(2\pi^2)^{\lambda}}\right)^{|\mathfrak{u}|}\right)^{1/(2\lambda)} \|F\|_{s,\boldsymbol{\gamma}} \qquad (3)$$

for all $\lambda \in (1/2, 1]$, where $F \in H_{s,\boldsymbol{\gamma}}$, $\zeta(x) := \sum_{k=1}^{\infty} k^{-x}$ is the Riemann zeta function for $x > 1$, $\mathbb{U}_n := \{k \in \mathbb{N} \mid 1 \leq k \leq n-1,\ \gcd(k,n) = 1\}$, and $\varphi(n) := |\mathbb{U}_n|$ is the Euler totient function.

Let $G\colon H_0^1(D) \to \mathbb{R}$ be a bounded linear functional and consider the function $f(\boldsymbol{y}) := G(u_s(\cdot, \boldsymbol{y} - \frac{\mathbf{1}}{\mathbf{2}})^2)$ for $\boldsymbol{y} \in [0,1]^s$ (**added 21.1.:** *alternatively, you may consider* $f(\boldsymbol{y}) := v(y_1 - \frac{1}{2}, \ldots, y_s - \frac{1}{2}, 0, 0, \ldots)$ *for* $\boldsymbol{y} \in [0,1]^s$*, where the function* v *was defined in* (1)). By plugging in the upper bound derived for $\|f\|_{s,\boldsymbol{\gamma}}$ in task 2 of last week's exercises (**added 21.1.:** *alternatively, use the bound* (2) *for* v), show that the bound in (3) can be minimized by choosing

$$\gamma_{\mathfrak{u}} := \left((|\mathfrak{u}|+1)! \prod_{j\in\mathfrak{u}} \frac{b_j}{\sqrt{2\zeta(2\lambda)/(2\pi^2)^{\lambda}}}\right)^{2/(1+\lambda)} \quad \text{for } \varnothing \neq \mathfrak{u} \subseteq \{1,\ldots,s\}, \qquad (4)$$

and $\gamma_{\varnothing} := 1$.

2. Suppose that n is prime and we use the CBC algorithm to obtain a generating vector using the weights (4). Consider $f(\boldsymbol{y}) := G(u_s(\cdot, \boldsymbol{y} - \frac{\mathbf{1}}{\mathbf{2}})^2)$ for $\boldsymbol{y} \in [0,1]^s$. (**Added 21.1.:** *alternatively, consider* $f(\boldsymbol{y}) := v(y_1 - \frac{1}{2}, \ldots, y_s - \frac{1}{2}, 0, 0, \ldots)$ *for* $\boldsymbol{y} \in [0,1]^s$*, where the function* v *was defined in* (1).) Show that the resulting randomly shifted rank-1 lattice rule satisfies the root-mean-square error bound

$$\sqrt{\mathbb{E}_{\boldsymbol{\Delta}}|I_sf - Q_{n,s}^{\boldsymbol{\Delta}}f|^2} \leq \begin{cases} Cn^{-1/p+1/2} & \text{if } p \in (2/3, 1), \\ Cn^{-1+\delta} \text{ for arbitrary } \delta \in (0, 1/2) & \text{if } p \in (0, 2/3], \end{cases} \qquad (5)$$

where the constant $C > 0$ is *independent* of the dimension s.

3. How would you choose the positive weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subseteq\{1,\ldots,s\}}$ to ensure a dimension-independent QMC cubature rate of the form (5) if you consider instead the function

$$f(\boldsymbol{y}) := \frac{1}{1 + \sum_{j=1}^{s} j^{-2} y_j^{\alpha}}, \quad \boldsymbol{y} := (y_1, \ldots, y_s) \in [0,1]^s,\ \alpha \in (0,1]?$$

4. For the dimension truncation proof, we actually need to consider a slightly more general variational formulation

$$\int_D a(\boldsymbol{x}, \boldsymbol{y}) \nabla u(\boldsymbol{x}, \boldsymbol{y}) \cdot \nabla v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \langle F, v\rangle_{H^{-1}(D), H_0^1(D)} \quad \text{for all } v \in H_0^1(D),$$

where $F \in H^{-1}(D)$ and the diffusion coefficient is defined as above and satisfies assumptions (A1)–(A3). By the Lax–Milgram lemma, there exists a unique solution $u(\cdot,\boldsymbol{y}) \in H_0^1(D)$ for all $\boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$ which satisfies the *a priori* bound $\|u(\cdot,\boldsymbol{y})\|_{H_0^1(D)} \leq \frac{\|F\|_{H^{-1}(D)}}{a_{\min}}$ for all $\boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$, where

$$\|F\|_{H^{-1}(D)} = \sup_{\substack{v \in H_0^1(D) \\ \|v\|_{H_0^1(D)} \leq 1}} |\langle F, v\rangle_{H^{-1}(D),H_0^1(D)}|.$$

(a) Let $a_s(\boldsymbol{x},\boldsymbol{y}) := a(\boldsymbol{x},(y_1,\ldots,y_s,0,0,\ldots))$ for $\boldsymbol{x} \in D$ and $\boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$. Prove the following version of the *second Strang lemma*:

$$\|u(\cdot,\boldsymbol{y}) - u_s(\cdot,\boldsymbol{y})\|_{H_0^1(D)} \leq \frac{1}{a_{\min}^2}\|a(\cdot,\boldsymbol{y}) - a_s(\cdot,\boldsymbol{y})\|_{L^\infty(D)}\|F\|_{H^{-1}(D)} \quad \text{for all } \boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}.$$

(b) Let $f(\boldsymbol{y}) := G(u(\cdot,\boldsymbol{y}))$, where $G \in H^{-1}(D)$ and $\boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$. Use part (a) to deduce that

$$\lim_{s\to\infty} f(y_1,\ldots,y_s,0,0,\ldots) = f(\boldsymbol{y}) \quad \text{for all } \boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$$

and

$$|f(y_1,\ldots,y_s,0,0,\ldots)| \leq |g(\boldsymbol{y})| \quad \text{for all } \boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}} \text{ and } s \in \mathbb{N}$$

for some integrable function $g\colon [-1/2,1/2]^{\mathbb{N}} \to \mathbb{R}$.

Hint: In part (a), take the difference of the variational formulations

$$\int_D a(\boldsymbol{x},\boldsymbol{y})\nabla u(\boldsymbol{x},\boldsymbol{y}) \cdot \nabla v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle F, v\rangle_{H^{-1}(D),H_0^1(D)} \quad \text{for all } v \in H_0^1(D),$$
$$\int_D a_s(\boldsymbol{x},\boldsymbol{y})\nabla u_s(\boldsymbol{x},\boldsymbol{y}) \cdot \nabla v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle F, v\rangle_{H^{-1}(D),H_0^1(D)} \quad \text{for all } v \in H_0^1(D),$$

and use the uniform ellipticity assumption (A1). Note that in part (b), we are essentially justifying the integrability of $f : [-1/2,1/2]^{\mathbb{N}} \to \mathbb{R}$. (See pg. 12 of the lecture notes of week 12.)