Uncertainty Quantification and Quasi-Monte Carlo Exercise 2
Wintersemester 2022/23
Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Tuesday 1 November, 2022, 12:15

1. Let H_1 and H_2 be real Hilbert spaces and let $A\colon H_1 \to H_2$ be a continuous linear operator. Recall that the *kernel* of operator A is defined as

$$\mathrm{Ker}(A) := \{x \in H_1 \mid Ax = 0\}$$

and the *range* of operator A is defined as

$$\mathrm{Ran}(A) := \{y \in H_2 \mid y = Ax \text{ for some } x \in H_1\}.$$

   (a) Show that $\mathrm{Ker}(A)$ is a *closed* subspace of H_1.
   (b) Show that $\mathrm{Ran}(A)$ is a subspace of H_2.
   (c) Show that $H_1 = \mathrm{Ker}(A) \oplus \overline{\mathrm{Ran}(A^*)}$ and $H_2 = \overline{\mathrm{Ran}(A)} \oplus \mathrm{Ker}(A^*)$, where the bar denotes the closure of a set and A^* is the adjoint of A.

2. (a) Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space and $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ is the induced norm. Prove the *polarization identity*

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \quad \text{for all } x, y \in X.$$

   (b) Let $(H_1, \langle\cdot,\cdot\rangle_{H_1})$ and $(H_2, \langle\cdot,\cdot\rangle_{H_2})$ be real Hilbert spaces. Suppose that $U\colon H_1 \to H_2$ is a linear *isometry*: $\|Ux\|_{H_2} = \|x\|_{H_1}$ for all $x \in H_1$, where $\|\cdot\|_{H_1} := \sqrt{\langle\cdot,\cdot\rangle}_{H_1}$ and $\|\cdot\|_{H_2} := \sqrt{\langle\cdot,\cdot\rangle}_{H_2}$. Show that

$$\langle Ux, Uy\rangle_{H_2} = \langle x, y\rangle_{H_1} \quad \text{for all } x, y \in H_1.$$

3. Let $D \subset \mathbb{R}^d$ be a measurable set. Recall that $L^2(D)$ is a Hilbert space when equipped with the inner product $\langle f, g\rangle_{L^2(D)} := \int_D f(\boldsymbol{x})g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, $f, g \in L^2(D)$.

   (a) Let $f \in L^2(D)$. Show that

$$\|f\|_{L^2(D)} = \sup_{\|g\|_{L^2(D)} \leq 1} \int_D f(\boldsymbol{x})g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

   (b) Show that for every $F \in (L^2(D))'$, there exists a *unique* element $f \in L^2(D)$ such that $\|F\|_{(L^2(D))'} = \|f\|_{L^2(D)}$ and

$$F(g) = \int_D f(\boldsymbol{x})g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{for all } g \in L^2(D).$$

4. Let H be a Hilbert space and assume that $A, B \in \mathcal{L}(H, H)$ commute, that is, $AB = BA$. If AB is invertible, what can you say about the invertibility of A and B?