Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Tuesday 8 November, 2022, 12:15

1. Recall that the *Fourier transform* of $u \in L^1(\mathbb{R}^d)$ is defined as

$$\widehat{u}(\boldsymbol{\xi}) := \mathcal{F}\{u\}(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}} u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{for } \boldsymbol{\xi} \in \mathbb{R}^d$$

and the *inverse Fourier transform* of $v \in L^1(\mathbb{R}^d)$ is defined as

$$\mathcal{F}^*\{v\}(\boldsymbol{x}) := \int_{\mathbb{R}^d} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}} v(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \quad \text{for } \boldsymbol{x} \in \mathbb{R}^d.$$

Consider the functions

$$\psi(\boldsymbol{x}) := \mathrm{e}^{-\pi\|\boldsymbol{x}\|^2} \quad \text{and} \quad \psi_\varepsilon(\boldsymbol{x}) = \varepsilon^{-d}\psi(\varepsilon^{-1}\boldsymbol{x}),$$

where $\boldsymbol{x} \in \mathbb{R}^d$ and $\varepsilon > 0$.

   (a) Show that
   $$\mathcal{F}\psi = \psi = \mathcal{F}^*\psi.$$

   (b) Show that
   $$\widehat{\psi_\varepsilon}(\boldsymbol{\xi}) = \widehat{\psi}(\varepsilon\boldsymbol{\xi}) \quad \text{and} \quad \mathcal{F}^*\widehat{\psi_\varepsilon} = \psi_\varepsilon.$$

2. Consider the *Schwartz space* of rapidly decreasing C^∞ functions:

$$\mathcal{S}(\mathbb{R}^d) := \left\{\varphi \in C^\infty(\mathbb{R}^d) \mid \sup_{\boldsymbol{x}\in\mathbb{R}^d} |\boldsymbol{x}^{\boldsymbol{\alpha}}\partial^{\boldsymbol{\beta}}\varphi(\boldsymbol{x})| < \infty \text{ for all } \boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{N}_0^d\right\}.$$

Show that[1]

$$\widehat{\partial^{\boldsymbol{\alpha}}\varphi}(\boldsymbol{\xi}) = (2\pi\mathrm{i}\boldsymbol{\xi})^{\boldsymbol{\alpha}}\widehat{\varphi}(\boldsymbol{\xi}) \quad \text{and} \quad \mathcal{F}\{(-2\pi\mathrm{i}\boldsymbol{x})^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x})\}(\boldsymbol{\xi}) = \partial^{\boldsymbol{\alpha}}\widehat{\varphi}(\boldsymbol{\xi})$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

3. Show that the function $f\colon x \mapsto |x|^{2/3}$ belongs to $H^1(-1,1)$ and that

$$f'(x) = c\,\mathrm{sign}(x)|x|^{-1/3}$$

for some constant c. Here $\mathrm{sign}(x) = -1$ if $x < 0$ and $\mathrm{sign}(x) = +1$ otherwise.

[1]Note that the *multi-index notations* used in this task are the following: $\boldsymbol{x}^{\boldsymbol{\alpha}} = \prod_{k=1}^d x_k^{\alpha_k}$, $\partial^{\boldsymbol{\beta}}\varphi(\boldsymbol{x}) = \big(\prod_{k=1}^d \frac{\partial^{\beta_k}}{\partial x_k^{\beta_k}}\big)\varphi(\boldsymbol{x})$, $(2\pi\mathrm{i}\boldsymbol{\xi})^{\boldsymbol{\alpha}} = \prod_{k=1}^d (2\pi\mathrm{i}\xi_k)^{\alpha_k}$, and $(-2\pi\mathrm{i}\boldsymbol{x})^{\boldsymbol{\alpha}} = \prod_{k=1}^d (-2\pi\mathrm{i}x_k)^{\alpha_k}$, where $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_d) \in \mathbb{N}_0^d$, $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_d) \in \mathbb{N}_0^d$, and $\boldsymbol{x} = (x_1,\ldots,x_d) \in \mathbb{R}^d$. We also use the convention $0^0 := 1$.