Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Tuesday 15 November, 2022, 12:15

1. Take a look at Definition 66, Corollary 74, and Exercise 76 in https://terrytao.wordpress.com/2010/10/16/245a-notes-5-differentiation-theorems/#lip-diff
   Show that any Lipschitz continuous function $f\colon \mathbb{R}\to\mathbb{R}$ is locally of bounded variation. Use Corollary 74 to conclude that f is differentiable almost everywhere.

2. Consider the *Dirichlet–Neumann problem*
$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}x}\left(a(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right)=1\quad\text{for } x\in(0,1),\\&u(0)=0,\\&u'(1)=0,\end{aligned}$$
   where $a\in C^1([0,1])$ is such that $a(x)\geq c>0$ for all $x\in[0,1]$. Show that
$$u(x)=\int_0^x\frac{1-y}{a(y)}\,\mathrm{d}y$$
   is a solution to this problem.

3. Show that the norms
$$\|u\|_{H^1(\mathbb{R}^d)}:=\left(\sum_{|\boldsymbol{\alpha}|\leq 1}\|\partial^{\boldsymbol{\alpha}}u\|_{L^2(\mathbb{R}^d)}^2\right)^{1/2},\quad u\in H^1(\mathbb{R}^d),$$
   and
$$\|u\|_{H^1(\mathbb{R}^d)}':=\left(\int_{\mathbb{R}^d}(1+\|\boldsymbol{\xi}\|^2)|\widehat{u}(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi}\right)^{1/2},\quad u\in H^1(\mathbb{R}^d),$$
   are equivalent. That is, there exist constants $c,C>0$ such that
$$c\|u\|_{H^1(\mathbb{R}^d)}'\leq\|u\|_{H^1(\mathbb{R}^d)}\leq C\|u\|_{H^1(\mathbb{R}^d)}'\quad\text{for all } u\in H^1(\mathbb{R}^d).$$
   Hint: Fourier transform turns differentiation into multiplication.

4. Show that for large enough dimension d, there holds
$$\frac{1}{\|\boldsymbol{x}\|}\in H^1(B(\mathbf{0},1)),$$
   where $\|\boldsymbol{x}\|=\sqrt{x_1^2+\cdots+x_d^2}$ for $\boldsymbol{x}=(x_1,\ldots,x_d)\in\mathbb{R}^d$ and $B(\mathbf{0},1)$ is the unit ball in $\mathbb{R}^d$.
   Hint: The change of variables formula for d-dimensional polar coordinates is
$$\int_{B(\mathbf{0},1)}f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_0^1\int_{\|\boldsymbol{\omega}\|=1}f(r\boldsymbol{\omega})r^{d-1}\,\mathrm{d}S(\boldsymbol{\omega})\,\mathrm{d}r,\qquad\text{(VK: typo fixed!)}$$
   where $C_d:=\int_{\|\boldsymbol{\omega}\|=1}\mathrm{d}S(\boldsymbol{\omega})=\frac{2\pi^{d/2}}{\Gamma(d/2)}$ and Γ denotes the gamma function.