

Return your written solutions either in person or by email
 to veska.kaarnioja@fu-berlin.de by Tuesday 22 November, 2022, 12:15

Please note that there are a total of 4 tasks this week!

1. Let $D \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded Lipschitz domain, $f \in L^2(D)$, and let $a \in L^\infty(D)$ be such that $0 < a_{\min} \leq a(\mathbf{x}) \leq a_{\max} < \infty$ for almost every $\mathbf{x} \in D$ for some constants $a_{\max}, a_{\min} > 0$. Let $u \in H_0^1(D)$ be the unique solution to the weak formulation

$$\int_D a(\mathbf{x}) \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) \, d\mathbf{x} = \int_D f(\mathbf{x}) v(\mathbf{x}) \, d\mathbf{x} \quad \text{for all } v \in H_0^1(D).$$

Let V_m be a finite dimensional subspace of $H_0^1(D)$ and consider the Galerkin solution $u_m \in V_m$ satisfying

$$\int_D a(\mathbf{x}) \nabla u_m(\mathbf{x}) \cdot \nabla v(\mathbf{x}) \, d\mathbf{x} = \int_D f(\mathbf{x}) v(\mathbf{x}) \, d\mathbf{x} \quad \text{for all } v \in V_m.$$

Show that

$$\|u - u_m\|_{H_0^1(D)} \leq \sqrt{\frac{a_{\max}}{a_{\min}}} \inf_{v \in V_m} \|u - v\|_{H_0^1(D)}.$$

Hint: First show that $\|a^{1/2} \nabla(u - u_m)\|_{L^2(D)}^2 = B(u - u_m, u - u_m) \stackrel{(*)}{\leq} B(u - v, u - v) = \|a^{1/2} \nabla(u - v)\|_{L^2(D)}^2$ for all $v \in V_m$, where $B(u, v) := \int_D a(\mathbf{x}) \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) \, d\mathbf{x}$. Modify the second step in the proof of Céa's lemma from the lecture notes of week 5 by upper bounding the term $B(u - u_m, u - u_m)$ using $(*)$.

2. Let $D \subset \mathbb{R}^2$ be a bounded polyhedron and let \mathcal{T}_h be a uniform, regular triangulation of D . Let $V_h = \text{span}(\phi_i)_{i=1}^m$ be a finite element subspace of $H_0^1(D)$ spanned by continuous, piecewise linear finite element basis functions such that $\phi_i(\mathbf{n}_j) = \delta_{i,j}$, where \mathbf{n}_i are vertices of the mesh elements lying in the interior of domain D and $\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. Let

$$u_h(\mathbf{x}) := \sum_{i=1}^m c_i \phi_i(\mathbf{x}), \quad c_1, \dots, c_m \in \mathbb{R},$$

be an element of the finite element space V_h and denote $\mathbf{c} := [c_1, \dots, c_m]^T$.

Show that

$$\|u_h\|_{L^2(D)} = \sqrt{\mathbf{c}^T M \mathbf{c}} \quad \text{and} \quad \|u_h\|_{H_0^1(D)} = \sqrt{\mathbf{c}^T S \mathbf{c}},$$

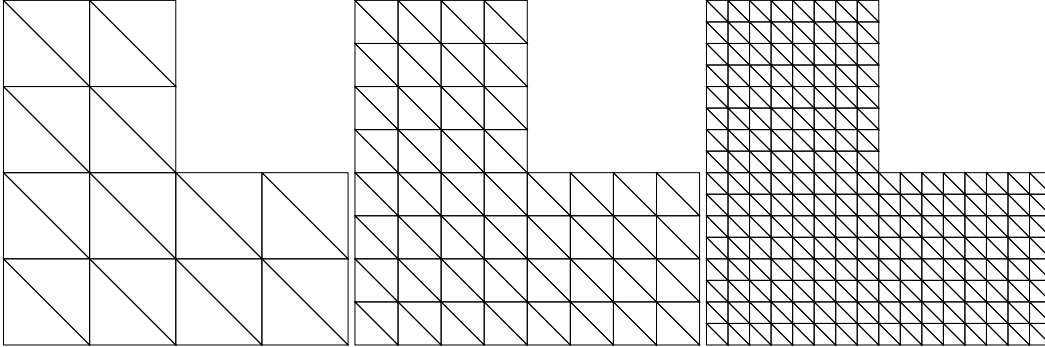
where $S := (S_{i,j})_{i,j=1}^m$ and $M := (M_{i,j})_{i,j=1}^m$ are defined by setting

$$S_{i,j} = \int_D \nabla \phi_i(\mathbf{x}) \cdot \nabla \phi_j(\mathbf{x}) \, d\mathbf{x} \quad \text{and} \quad M_{i,j} = \int_D \phi_i(\mathbf{x}) \phi_j(\mathbf{x}) \, d\mathbf{x}.$$

Use this to conclude that S and M are *positive definite* matrices: that is, $\mathbf{x}^T S \mathbf{x} > 0$ and $\mathbf{x}^T M \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$.

The exercises continue on the next page!

3. Let $D := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 \leq 1, 0 < x_2 < 2\} \cup \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 \leq x_1 < 2, 0 < x_2 < 1\} \subset \mathbb{R}^2$ be an *L-shaped domain*. Modify the program `FEMdata.m` on the course page¹ to create a uniform, regular triangulation \mathcal{T}_h of the L-shaped domain with mesh widths $h \in \{2^{-1}, 2^{-2}, 2^{-3}, \dots\}$. Try also plotting your triangulations. The goal is to obtain something like these triangulations:



Please make sure to return the source code you used to solve this task.

4. Let $D \subset \mathbb{R}^2$ be a bounded polyhedron. Let us consider the *spectral eigenvalue problem* of finding the eigenvalues $\lambda \in \mathbb{R}$ and eigenfunctions $u : D \rightarrow \mathbb{R}$ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } D, \\ u|_{\partial D} = 0, \\ \int_D u(\mathbf{x})^2 d\mathbf{x} = 1. \end{cases}$$

The weak formulation of this problem is to find $(\lambda, u) \in \mathbb{R} \times (H_0^1(D) \setminus \{\mathbf{0}\})$, $\|u\|_{L^2(D)} = 1$, such that

$$\int_D \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} = \lambda \int_D u(\mathbf{x})v(\mathbf{x}) d\mathbf{x} \quad \text{for all } v \in H_0^1(D).$$

If V_m is a finite dimensional subspace of $H_0^1(D)$, then the goal is to find $(\lambda, u_m) \in \mathbb{R} \times (V_m \setminus \{\mathbf{0}\})$, $\|u_m\|_{L^2(D)} = 1$, such that

$$\int_D \nabla u_m(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} = \lambda \int_D u_m(\mathbf{x})v(\mathbf{x}) d\mathbf{x} \quad \text{for all } v \in V_m. \quad (1)$$

- (a) Let $V_h = \text{span}(\phi_i)_{i=1}^m$ be a finite element subspace of $H_0^1(D)$ spanned by continuous, piecewise linear finite element basis functions such that $\phi_i(\mathbf{n}_j) = \delta_{i,j}$, where \mathbf{n}_i are vertices of the mesh elements lying in the interior of the domain D . Show that (1) can be solved by considering the *generalized eigenvalue problem*

$$S\mathbf{c} = \lambda M\mathbf{c}, \quad \mathbf{c}^T M\mathbf{c} = 1, \quad (2)$$

where $\mathbf{c} := [c_1, \dots, c_m]^T$ are the finite element expansion coefficients of the corresponding finite element discretized eigenfunction $u_h(\mathbf{x}) = \sum_{i=1}^m c_i \phi_i(\mathbf{x})$ and $S = (S_{i,j})_{i,j=1}^m$ and $M = (M_{i,j})_{i,j=1}^m$ are defined by the formulae $S_{i,j} = \int_D \nabla \phi_i(\mathbf{x}) \cdot \nabla \phi_j(\mathbf{x}) d\mathbf{x}$ and $M_{i,j} = \int_D \phi_i(\mathbf{x})\phi_j(\mathbf{x}) d\mathbf{x}$.

The exercises continue on the next page!

¹Or write your own implementation using your favorite programming language! For this task, it's enough to reproduce the finite element vertex array `nodes` and mesh element connectivity array `element` appearing in `FEMdata.m` for the L-shaped domain.

- (b) Download the file `week5.mat` from the course website.

MATLAB users: You can import the contents of the file to MATLAB using the command `load week5`; The file contains the objects `mass`, `stiffness`, `nodes`, `element`, `interior`, `centers`, `ncoord`, and `nelem`. These correspond to the mass matrix and stiffness matrix, arrays containing the finite element nodes and mesh element connectivities, the indices of the interior coordinates, element center points, and the number of finite element nodes and elements, respectively. In this case, the computational domain D is the L-shaped domain considered in task 3.

Please note that the zero Dirichlet boundary condition has already been imposed on the matrices `mass` and `stiffness`.

Your task is to find the *smallest eigenpair* satisfying (2) and plot the corresponding eigenfunction.

In MATLAB, it is probably the most convenient to solve the smallest eigenpair of a generalized eigenvalue problem using a command like

```
[evecs,evals] = eigs(stiffness,mass,1,'smallestabs');    (*)
```

Python users: You will need to have (at least) `numpy` and `scipy` packages installed. This can be done, e.g., via `pip install numpy` and `pip install scipy` (or `pip3 install numpy` and `pip3 install scipy` depending on your configuration). In the preamble, you will require

```
import numpy
import scipy.io
from scipy.sparse.linalg import eigsh
```

The file can be imported using the command

```
mat = scipy.io.loadmat('week5.mat')
```

and you can access the objects by calling `mat['mass']`, `mat['stiffness']`, `mat['nodes']`, etc. (See the detailed description of file contents in the MATLAB section above.) The analogue of the MATLAB command (*) is `evals,evecs = eigsh(mat['stiffness'],k=1,M=mat['mass'],which='SM')`

Your task is to find the *smallest eigenpair* satisfying (2) and plot the corresponding eigenfunction. For visualizing your results, you can use, e.g., `matplotlib` (installation again via `pip install matplotlib` or `pip3 install matplotlib`).

MATLAB and Python users: Please make sure to return the source code you used to solve the task. *Hint:* the eigenfunction should look like the function below:

