Uncertainty Quantification and Quasi-Monte Carlo — Exercise 7
Sommersemester 2025
Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Tuesday 17 May 2025, 10:15 am

1. Recall from the lectures that, for given dimension $s \geq 1$, number of QMC nodes $n \geq 2$, and sequence of positive weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subseteq \{1,\ldots,s\}}$, the generating vector $\boldsymbol{z} \in \mathbb{U}_n^s$ obtained using the CBC algorithm satisfies the error bound

$$\sqrt{\mathbb{E}_{\boldsymbol{\Delta}}|I_s F - Q_{n,s}^{\boldsymbol{\Delta}} F|^2} \leq \bigg(\frac{1}{\varphi(n)} \sum_{\varnothing \neq \mathfrak{u} \subseteq \{1,\ldots,s\}} \gamma_{\mathfrak{u}}^{\lambda} \bigg(\frac{2\zeta(2\lambda)}{(2\pi^2)^{\lambda}}\bigg)^{|\mathfrak{u}|}\bigg)^{1/(2\lambda)} \|F\|_{s,\boldsymbol{\gamma}} \quad (1)$$

for all $\lambda \in (1/2, 1]$, where $F \in H_{s,\boldsymbol{\gamma}}$, $\zeta(x) := \sum_{k=1}^{\infty} k^{-x}$ is the Riemann zeta function for $x > 1$, $\mathbb{U}_n := \{k \in \mathbb{N} \mid 1 \leq k \leq n,\ \gcd(k,n) = 1\}$, and $\varphi(n) := |\mathbb{U}_n|$ is the Euler totient function.

Consider the function

$$F(\boldsymbol{y}) := \frac{1}{1 + \sum_{j=1}^{s} j^{-2} y_j^{\alpha}}, \quad \boldsymbol{y} := (y_1, \ldots, y_s) \in [0,1]^s,\ \alpha \in (0,1].$$

Explain, how would you choose the positive weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subseteq \{1,\ldots,s\}}$ to ensure a dimension-independent QMC cubature rate of the form

$$\sqrt{\mathbb{E}_{\boldsymbol{\Delta}}|I_s F - Q_{n,s}^{\boldsymbol{\Delta}} F|^2} \leq \begin{cases} C n^{-1/p + 1/2} & \text{if } p \in (2/3, 1), \\ C n^{-1+\delta} \text{ for arbitrary } \delta \in (0, 1/2) & \text{if } p \in (0, 2/3]. \end{cases}$$

2. Let $D \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, be a nonempty, bounded Lipschitz domain. Let us consider the variational formulation

$$\int_D a(\boldsymbol{x}, \boldsymbol{y}) \nabla u(\boldsymbol{x}, \boldsymbol{y}) \cdot \nabla v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \langle F, v \rangle_{H^{-1}(D), H_0^1(D)} \quad \text{for all } v \in H_0^1(D),$$

where $F \in H^{-1}(D)$, the diffusion coefficient is defined as

$$a(\boldsymbol{x}, \boldsymbol{y}) = a_0(\boldsymbol{x}) + \sum_{j=1}^{\infty} y_j \psi_j(\boldsymbol{x}), \quad \boldsymbol{x} \in D,\ \boldsymbol{y} = (y_j)_{j \geq 1} \in [-1/2, 1/2]^{\mathbb{N}},$$

and it satisfies the following:

(A1) There exist constants $a_{\min}, a_{\max} > 0$ such that $0 < a_{\min} \leq a(\boldsymbol{x}, \boldsymbol{y}) \leq a_{\max} < \infty$ for all $\boldsymbol{x} \in D$ and $\boldsymbol{y} \in [-1/2, 1/2]^{\mathbb{N}}$.

(A2) $a_0 \in L^{\infty}(D)$ and $\psi_j \in L^{\infty}(D)$ for all $j \geq 1$ such that $\sum_{j \geq 1} \|\psi_j\|_{L^{\infty}(D)} < \infty$.

(A3) For some $p \in (0,1)$, there holds $\sum_{j \geq 1} \|\psi_j\|_{L^{\infty}(D)}^p < \infty$.

By the Lax–Milgram lemma, there exists a unique solution $u(\cdot,\boldsymbol{y}) \in H_0^1(D)$ for all $\boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$ which satisfies the *a priori* bound $\|u(\cdot,\boldsymbol{y})\|_{H_0^1(D)} \leq \frac{\|F\|_{H^{-1}(D)}}{a_{\min}}$ for all $\boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$, where

$$\|F\|_{H^{-1}(D)} = \sup_{\substack{v \in H_0^1(D) \\ \|v\|_{H_0^1(D)} \leq 1}} |\langle F, v\rangle_{H^{-1}(D),H_0^1(D)}|.$$

(a) Let $a_s(\boldsymbol{x},\boldsymbol{y}) := a(\boldsymbol{x},(y_1,\ldots,y_s,0,0,\ldots))$ for $\boldsymbol{x} \in D$ and $\boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$. Prove the following version of the *second Strang lemma*:

$$\|u(\cdot,\boldsymbol{y}) - u_s(\cdot,\boldsymbol{y})\|_{H_0^1(D)} \leq \frac{1}{a_{\min}^2}\|a(\cdot,\boldsymbol{y}) - a_s(\cdot,\boldsymbol{y})\|_{L^\infty(D)}\|F\|_{H^{-1}(D)} \quad \text{for all } \boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}.$$

(b) Let $f(\boldsymbol{y}) := G(u(\cdot,\boldsymbol{y}))$, where $G \in H^{-1}(D)$ and $\boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$. Use part (a) to deduce that

$$\lim_{s\to\infty} f(y_1,\ldots,y_s,0,0,\ldots) = f(\boldsymbol{y}) \quad \text{for all } \boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$$

and

$$|f(y_1,\ldots,y_s,0,0,\ldots)| \leq |g(\boldsymbol{y})| \quad \text{for all } \boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}} \text{ and } s \in \mathbb{N}$$

for some integrable function $g\colon [-1/2,1/2]^{\mathbb{N}} \to \mathbb{R}$.

Hint: In part (a), take the difference of the variational formulations

$$\int_D a(\boldsymbol{x},\boldsymbol{y})\nabla u(\boldsymbol{x},\boldsymbol{y}) \cdot \nabla v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle F, v\rangle_{H^{-1}(D),H_0^1(D)} \quad \text{for all } v \in H_0^1(D),$$
$$\int_D a_s(\boldsymbol{x},\boldsymbol{y})\nabla u_s(\boldsymbol{x},\boldsymbol{y}) \cdot \nabla v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle F, v\rangle_{H^{-1}(D),H_0^1(D)} \quad \text{for all } v \in H_0^1(D),$$

and use the uniform ellipticity assumption (A1). Note that in part (b), we are essentially justifying the integrability of $f\colon [-1/2,1/2]^{\mathbb{N}} \to \mathbb{R}$.

3. Let $\mathscr{F} := \{\boldsymbol{m} \in \mathbb{N}_0^{\mathbb{N}} \mid |\boldsymbol{m}| := \sum_{j=1}^\infty m_j < \infty\}$ denote the set of finitely supported multi-indices. Let $\boldsymbol{b} = (b_j)_{j\geq 1}$ be a sequence of non-negative numbers and let $(\mathbb{A}_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathscr{F}}$ and $(\mathbb{B}_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathscr{F}}$ be non-negative numbers satisfying the inequality

$$\mathbb{A}_{\boldsymbol{\nu}} \leq \sum_{j\in\mathrm{supp}(\boldsymbol{\nu})} \nu_j b_j \mathbb{A}_{\boldsymbol{\nu}-\boldsymbol{e}_j} + \mathbb{B}_{\boldsymbol{\nu}} \quad \text{for all } \boldsymbol{\nu} \in \mathscr{F} \text{ (including } \boldsymbol{\nu} = \boldsymbol{0}),$$

where $\mathrm{supp}(\boldsymbol{\nu}) := \{j \in \mathbb{N} \mid \nu_j \neq 0\}$. Prove that

$$\mathbb{A}_{\boldsymbol{\nu}} \leq \sum_{\boldsymbol{m}\leq\boldsymbol{\nu}} \binom{\boldsymbol{\nu}}{\boldsymbol{m}} |\boldsymbol{m}|!\,\boldsymbol{b}^{\boldsymbol{m}}\mathbb{B}_{\boldsymbol{\nu}-\boldsymbol{m}} \quad \text{for all } \boldsymbol{\nu} \in \mathscr{F}.$$

4. Suppose that $F\colon [-1/2,1/2]^{\mathbb{N}} \to \mathbb{R}$ is an infinitely many times continuously differentiable function such that

$$|\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y})| \leq C(|\boldsymbol{\nu}|!)^\beta \boldsymbol{b}^{\boldsymbol{\nu}} \quad \text{for all } \boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}} \text{ and } \boldsymbol{\nu} \in \mathscr{F},$$

where $C, \beta \geq 1$ are constants and $\boldsymbol{b} = (b_j)_{j\geq 1}$ is a sequence of nonnegative real numbers. Show that

$$|\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F(\boldsymbol{y})^2| \leq C^2((|\boldsymbol{\nu}|+1)!)^\beta \boldsymbol{b}^{\boldsymbol{\nu}} \quad \text{for all } \boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}} \text{ and } \boldsymbol{\nu} \in \mathscr{F}.$$