Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Tuesday 10 January, 2023, 12:15

1. Let $s,n\in\mathbb{N}$, $z_1,\ldots,z_s\in\mathbb{U}_n:=\{k\in\mathbb{N}\mid 1\leq k\leq n-1,\ \gcd(k,n)=1\}$, and $\gamma_{\mathfrak{u}}\in\mathbb{R}_+$ for all $\varnothing\neq\mathfrak{u}\subseteq\{1,\ldots,s\}$. During the lectures, we derived the following formula for the *shift-averaged worst-case error* for integrands belonging to unanchored, weighted Sobolev spaces:

$$[e_{n,s}^{\rm sh}(z_1,\ldots,z_s)]^2=\frac{1}{n}\sum_{\varnothing\neq\mathfrak{u}\subseteq\{1,\ldots,s\}}\gamma_{\mathfrak{u}}\sum_{k=0}^{n-1}\prod_{j\in\mathfrak{u}}B_2\bigg(\bigg\{\frac{kz_j}{n}\bigg\}\bigg),$$

where the braces $\{x\}:=x-\lfloor x\rfloor$ denote the *fractional part* of a non-negative real number $x\geq 0$, $\lfloor x\rfloor:=\max\{k\in\mathbb{Z}:k\leq x\}$ for $x\in\mathbb{R}$, and $B_2(x):=x^2-x+\frac{1}{6}$ is the Bernoulli polynomial of degree 2.

   (a) When $s=1$, show that

$$[e_{n,1}^{\rm sh}(z_1)]^2=\frac{\gamma_{\{1\}}}{6n^2}.$$

   (b) Use part (a) to conclude that

$$[e_{n,1}^{\rm sh}(z_1)]^2\leq\bigg(\frac{1}{\varphi(n)}\gamma_{\{1\}}^{\lambda}\frac{2\zeta(2\lambda)}{(2\pi^2)^{\lambda}}\bigg)^{1/\lambda}\quad\text{for all }\lambda\in\big(\tfrac{1}{2},1\big],$$

   where $\varphi(n):=|\mathbb{U}_n|$ is the *Euler totient function*[1] for $n\in\mathbb{N}$ and $\zeta(x):=\sum_{k=1}^{\infty}k^{-x}$ is the *Riemann zeta function* for $x>1$.

2. Take a look at Subsection 1.5: "A new kind of example" in the *open access* article `https://doi.org/10.1017/S1446181112000077`

   The authors consider the function

$$F(\boldsymbol{y}):=\frac{1}{1+\sum_{j=1}^{s}j^{-2}y_j^{\alpha}},\quad \boldsymbol{y}:=(y_1,\ldots,y_s)\in[0,1]^s,\quad \alpha\in(0,1].$$

   (a) Verify the authors' claim that

$$\frac{\partial^{|\mathfrak{u}|}}{\partial\boldsymbol{y}_{\mathfrak{u}}}F(\boldsymbol{y})=|\mathfrak{u}|!F(\boldsymbol{y})^{|\mathfrak{u}|+1}\prod_{j\in\mathfrak{u}}(-\alpha j^{-2}y_j^{\alpha-1})\quad\text{for all }\varnothing\neq\mathfrak{u}\subseteq\{1,\ldots,s\}.$$

   Note that here $\frac{\partial^{|\mathfrak{u}|}}{\partial\boldsymbol{y}_{\mathfrak{u}}}=\prod_{j\in\mathfrak{u}}\frac{\partial}{\partial y_j}$ for all $\varnothing\neq\mathfrak{u}\subseteq\{1,\ldots,s\}$ and $|\mathfrak{u}|$ denotes the cardinality of set $\mathfrak{u}\subseteq\{1,\ldots,s\}$.

The exercises continue on the next page!

[1]Note that the bars $|\cdot|$ in the definition are used to denote the cardinality of a set.

(b) Let $\boldsymbol{\gamma} := (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subseteq\{1,\ldots,s\}}$ be a sequence of positive real numbers. During the lectures we considered an unanchored, weighted Sobolev space $H_{s,\boldsymbol{\gamma}}$ equipped with the norm

$$\|f\|_{s,\boldsymbol{\gamma}}^2 = \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}} \frac{1}{\gamma_{\mathfrak{u}}} \int_{[0,1]^{|\mathfrak{u}|}} \bigg(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{y}_{\mathfrak{u}}} f(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}_{-\mathfrak{u}} \bigg)^2 \mathrm{d}\boldsymbol{y}_{\mathfrak{u}}, \quad f \in H_{s,\boldsymbol{\gamma}},$$

where $\mathrm{d}\boldsymbol{y}_{\mathfrak{u}} := \prod_{j\in\mathfrak{u}} \mathrm{d}y_j$ and $\mathrm{d}\boldsymbol{y}_{-\mathfrak{u}} := \prod_{j\in\{1,\ldots,s\}\setminus\mathfrak{u}} \mathrm{d}y_j$ for $\mathfrak{u} \subseteq \{1,\ldots,s\}$. Show that the function considered in part (a) satisfies

$$\|F\|_{s,\boldsymbol{\gamma}}^2 \leq \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}} \frac{(|\mathfrak{u}|!)^2 \prod_{j\in\mathfrak{u}} b_j^2}{\gamma_{\mathfrak{u}}}, \quad \text{where } b_j = \frac{\alpha}{j^2\sqrt{2\alpha-1}}.$$

Hint: Subsection 1.5 in the aforementioned article may be helpful! Reader beware: the authors use a *different* definition for the norm $\|\cdot\|_{s,\boldsymbol{\gamma}}$ than we do!

3. Let $s \in \mathbb{N}$, let $f, g\colon \mathbb{R}^s \to \mathbb{R}$ be infinitely many times continuously differentiable, and let $\boldsymbol{\nu} \in \mathbb{N}_0^s$ be a multi-index. Show that

$$\partial^{\boldsymbol{\nu}}(f(\boldsymbol{y})g(\boldsymbol{y})) = \sum_{\boldsymbol{m}\leq\boldsymbol{\nu}} \binom{\boldsymbol{\nu}}{\boldsymbol{m}} \partial^{\boldsymbol{m}} f(\boldsymbol{y}) \partial^{\boldsymbol{\nu}-\boldsymbol{m}} g(\boldsymbol{y}) \quad \text{for all } \boldsymbol{y} \in \mathbb{R}^s.$$

Here, we have used the following multi-index notations:

$$\begin{aligned} \boldsymbol{\nu} &= (\nu_1,\ldots,\nu_s) \in \mathbb{N}_0^s \\ \boldsymbol{m} &= (m_1,\ldots,m_s) \in \mathbb{N}_0^s \\ \boldsymbol{m} \leq \boldsymbol{\nu} &\Leftrightarrow m_j \leq \nu_j \text{ for all } j \in \{1,\ldots,s\} \\ \partial^{\boldsymbol{\nu}} &= \prod_{j=1}^s \frac{\partial^{\nu_j}}{\partial y_j^{\nu_j}} \\ \binom{\boldsymbol{\nu}}{\boldsymbol{m}} &= \prod_{j=1}^s \binom{\nu_j}{m_j}. \end{aligned}$$

Hint: Use induction with respect to the order of multi-index $\boldsymbol{\nu} \in \mathbb{N}_0^s$. Pages 13–14 in `https://vesak90.userpage.fu-berlin.de/week11.pdf` may be helpful.

4. (a) Let $s \in \mathbb{N}$ and $\gamma_j \geq 0$ for all $j \in \{1,\ldots,s\}$. Show that

$$\sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}} \prod_{j\in\mathfrak{u}} \gamma_j = \prod_{j=1}^s (1+\gamma_j).$$

(b) Let $\mathfrak{u} \subseteq \{1,\ldots,s\}$. Show that

$$|\mathfrak{u}|! \leq \prod_{j\in\mathfrak{u}} j.$$

(c) Let $\ell \in \mathbb{N}_0$ and $c_j \geq 0$ for all $j \in \mathbb{N}$. Show that

$$\sum_{\substack{|\mathfrak{u}|=\ell \\ \mathfrak{u}\subseteq\mathbb{N}}} \prod_{j\in\mathfrak{u}} c_j \leq \frac{1}{\ell!} \bigg(\sum_{j=1}^\infty c_j \bigg)^\ell.$$