Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Tuesday 17 January, 2023, 12:15

In this week's exercises, we consider an elliptic PDE problem. Let $D \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, be a bounded Lipschitz domain and let $f \in L^2(D)$ be fixed. For all $\boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$, let $u(\cdot,\boldsymbol{y}) \in H_0^1(D)$ be such that

$$\int_D a(\boldsymbol{x},\boldsymbol{y})\nabla u(\boldsymbol{x},\boldsymbol{y})\cdot\nabla v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_D f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{for all } v \in H_0^1(D), \tag{1}$$

with the diffusion coefficient

$$a(\boldsymbol{x},\boldsymbol{y}) = a_0(\boldsymbol{x}) + \sum_{j=1}^{\infty} y_j\psi_j(\boldsymbol{x}), \quad \boldsymbol{x} \in D,\ \boldsymbol{y} = (y_j)_{j\geq 1} \in [-1/2,1/2]^{\mathbb{N}},$$

where we assume the following:

(A1) There exist constants $a_{\min}, a_{\max} > 0$ such that $0 < a_{\min} \leq a(\boldsymbol{x},\boldsymbol{y}) \leq a_{\max} < \infty$ for all $\boldsymbol{x} \in D$ and $\boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}$.

(A2) $a_0 \in L^\infty(D)$ and $\psi_j \in L^\infty(D)$ for all $j \geq 1$ such that $\sum_{j\geq 1}\|\psi_j\|_{L^\infty(D)} < \infty$.

(A3) For some $p \in (0,1)$, there holds $\sum_{j\geq 1}\|\psi_j\|_{L^\infty(D)}^p < \infty$.

Moreover, we define the dimensionally-truncated solution by setting $u_s(\cdot,(y_1,\ldots,y_s)) := u(\cdot,(y_1,\ldots,y_s,0,0,\ldots)$ for $y_j \in [-1/2,1/2]$, $1 \leq j \leq s$.

Added 21.1.: The stated regularity bound for $\partial^{\boldsymbol{\nu}} u(\cdot,\boldsymbol{y})^2$ in task 2 contains an unfortunate error. Similar results are true *mutatis mutandis* by substituting in place of $u(\cdot,\boldsymbol{y})^2$ the function

$$v(\boldsymbol{y}) := \int_D u(\boldsymbol{x},\boldsymbol{y})^2\,\mathrm{d}\boldsymbol{x}, \quad \boldsymbol{y} \in [-1/2,1/2]^{\mathbb{N}}.$$

For further details, please see
`https://vesak90.userpage.fu-berlin.de/erratum.PNG`

Your tasks are outlined below.

1. Let $\mathscr{F} := \left\{\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}} : |\boldsymbol{\nu}| := \sum_{j\geq 1}\nu_j < \infty\right\}$ denote the set of finitely supported multi-indices. Prove the *generalized Vandermonde identity*

$$\sum_{\substack{|\boldsymbol{m}|=\ell \\ \boldsymbol{m}\leq\boldsymbol{\nu}}} \binom{\boldsymbol{\nu}}{\boldsymbol{m}} = \binom{|\boldsymbol{\nu}|}{\ell} \quad \text{for all } \boldsymbol{\nu} \in \mathscr{F} \text{ and } 0 \leq \ell \leq |\boldsymbol{\nu}|.$$

   Hint: You can use a simple combinatorial argument or induction with respect to the order of the multi-indices.

2. For some constant $C > 0$, show that the *squared* solution to (1) satisfies the regularity bound

$$\|\partial^{\boldsymbol{\nu}} u(\cdot,\boldsymbol{y})^2\|_{H_0^1(D)} \leq C(|\boldsymbol{\nu}|+1)!\boldsymbol{b}^{\boldsymbol{\nu}} \quad \text{for all } \boldsymbol{\nu}\in\mathscr{F} \text{ and } \boldsymbol{y}\in[-1/2,1/2]^{\mathbb{N}},$$

$$|\partial^{\boldsymbol{\nu}} v(\boldsymbol{y})| \leq C(|\boldsymbol{\nu}|+1)!\boldsymbol{b}^{\boldsymbol{\nu}} \quad \text{for all } \boldsymbol{\nu}\in\mathscr{F} \text{ and } \boldsymbol{y}\in[-1/2,1/2]^{\mathbb{N}},$$

where $\boldsymbol{b} := (b_j)_{j\geq 1}$ is defined by setting $b_j := \frac{\|\psi_j\|_{L^\infty(D)}}{a_{\min}}$ for all $j \geq 1$.

Hint: Use the Leibniz product rule (see task 3 of exercise 8), the regularity bound for $\|\partial^{\boldsymbol{\nu}} u(\cdot,\boldsymbol{y})\|_{H_0^1(D)}$ proved during the lecture, and the generalized Vandermonde identity proved in task 1. Note that $\sum_{\boldsymbol{m}\leq\boldsymbol{\nu}} = \sum_{\ell=0}^{|\boldsymbol{\nu}|}\sum_{\substack{|\boldsymbol{m}|=\ell\\ \boldsymbol{m}\leq\boldsymbol{\nu}}}$.

3. Let $s \geq 1$ be the dimension, let $\boldsymbol{\gamma} := (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subseteq\{1,\ldots,s\}}$ be a sequence of positive weights, and consider the weighted, unanchored Sobolev space $H_{s,\boldsymbol{\gamma}}$ equipped with the norm

$$\|F\|_{s,\boldsymbol{\gamma}}^2 = \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}} \frac{1}{\gamma_{\mathfrak{u}}} \int_{[0,1]^{|\mathfrak{u}|}} \bigg(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{y}_{\mathfrak{u}}} F(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}_{-\mathfrak{u}}\bigg)^2 \mathrm{d}\boldsymbol{y}_{\mathfrak{u}}, \quad F\in H_{s,\boldsymbol{\gamma}},$$

where $\mathrm{d}\boldsymbol{y}_{\mathfrak{u}} := \prod_{j\in\mathfrak{u}} \mathrm{d}y_j$ and $\mathrm{d}\boldsymbol{y}_{-\mathfrak{u}} := \prod_{j\in\{1,\ldots,s\}\setminus\mathfrak{u}} \mathrm{d}y_j$.

Let $G : H_0^1(D) \to \mathbb{R}$ be a bounded linear functional and define $f(\boldsymbol{y}) := G(u_s(\cdot,\boldsymbol{y}-\frac{\mathbf{1}}{\mathbf{2}})^2)$ for $\boldsymbol{y}\in[0,1]^s$. Using task 2, show that

$$\|f\|_{s,\boldsymbol{\gamma}}^2 \leq C^2\|G\|_{H_0^1(D)\to\mathbb{R}}^2 \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}} \frac{[(|\mathfrak{u}|+1)!]^2}{\gamma_{\mathfrak{u}}} \prod_{j\in\mathfrak{u}} b_j^2.$$

Let $g(\boldsymbol{y}) := g(y_1-\frac{1}{2},\ldots,y_s-\frac{1}{2},0,0,\ldots)$ for $\boldsymbol{y}\in[0,1]^s$. Then

$$\|g\|_{s,\boldsymbol{\gamma}}^2 \lesssim \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}} \frac{[(|\mathfrak{u}|+1)!]^2}{\gamma_{\mathfrak{u}}} \prod_{j\in\mathfrak{u}} b_j^2.$$

4. **Added 21.1.: This is also wrong.** Let $k\in\mathbb{N}$. For some constant $C > 0$ (which may depend on k), show that

$$\|\partial^{\boldsymbol{\nu}} u(\cdot,\boldsymbol{y})^k\|_{H_0^1(D)} \leq C(|\boldsymbol{\nu}|+k-1)!\boldsymbol{b}^{\boldsymbol{\nu}} \quad \text{for all } \boldsymbol{\nu}\in\mathscr{F} \text{ and } \boldsymbol{y}\in[-1/2,1/2]^{\mathbb{N}},$$

where $\boldsymbol{b} := (b_j)_{j\geq 1}$ is defined as in task 2.

Let $s \geq 1$ be the dimension, $\boldsymbol{\gamma} := (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subseteq\{1,\ldots,s\}}$ a sequence of positive weights, $G\colon H_0^1(D)\to\mathbb{R}$ a bounded linear functional, and let $f_k(\boldsymbol{y}) := G(u_s(\cdot,\boldsymbol{y}-\frac{\mathbf{1}}{\mathbf{2}})^k)$ for $k\in\mathbb{N}$ and $\boldsymbol{y}\in[0,1]^s$. Show that

$$\|f_k\|_{s,\boldsymbol{\gamma}}^2 \leq C^2\|G\|_{H_0^1(D)\to\mathbb{R}}^2 \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}} \frac{[(|\mathfrak{u}|+k-1)!]^2}{\gamma_{\mathfrak{u}}} \prod_{j\in\mathfrak{u}} b_j^2.$$