Inverse Problems — Exercise 1

Sommersemester 2023

Return your written solutions either in person or by email to vesa.kaarnioja@fu-berlin.de by Tuesday 25 April, 2023, 10:15

1. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and $\| \cdot \|$ is the induced norm.
   (a) Prove that an inner product satisfies the *parallelogram identity*
   $$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all } x, y \in X.$$
   (b) Prove the *polarization identity*
   $$\langle x, y \rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \quad \text{for all } x, y \in X.$$
   (c) Let $(H_1, \langle \cdot, \cdot \rangle_{H_1})$ and $(H_2, \langle \cdot, \cdot \rangle_{H_2})$ be real Hilbert spaces. Suppose that $U\colon H_1 \to H_2$ is a linear *isometry*: $\|Ux\|_{H_2} = \|x\|_{H_1}$ for all $x \in H_1$, where $\| \cdot \|_{H_1} := \sqrt{\langle \cdot, \cdot \rangle}_{H_1}$ and $\| \cdot \|_{H_2} := \sqrt{\langle \cdot, \cdot \rangle}_{H_2}$. Show that
   $$\langle Ux, Uy \rangle_{H_2} = \langle x, y \rangle_{H_1} \quad \text{for all } x, y \in H_1.$$
2. Let $(X_i, \| \cdot \|_i)$ be normed spaces, $i \in \{1, 2, 3\}$, and $X_1 \neq \{0\}$. Show that for the norm of a linear operator $A\colon X_1 \to X_2$, we have
   $$\|A\| = \sup_{0 \neq \|x\|_1 \leq 1} \frac{\|Ax\|_2}{\|x\|_1} = \sup_{\|x\|_1 = 1} \frac{\|Ax\|_2}{\|x\|_1}.$$
   Let also $B\colon X_2 \to X_3$ be linear. Prove that
   $$\|BA\| \leq \|B\| \|A\|.$$
3. Let H_1 and H_2 be real Hilbert spaces and let $A\colon H_1 \to H_2$ be a continuous linear operator. Recall that the *kernel* of operator A is defined as
   $$\mathrm{Ker}(A) := \{x \in H_1 \mid Ax = 0\}$$
   and the *range* of operator A is defined as
   $$\mathrm{Ran}(A) := \{y \in H_2 \mid y = Ax \text{ for some } x \in H_1\}.$$
   (a) Show that $\mathrm{Ker}(A)$ is a *closed* subspace of H_1.
   (b) Show that $\mathrm{Ran}(A)$ is a subspace of H_2.
4. Let H be a real Hilbert space. Recall that the *orthogonal complement* of any subset $M \subset H$ is defined as
   $$M^\perp := \{y \in H \mid \langle x, y \rangle = 0 \text{ for all } x \in M\}.$$
   (a) Show that for any subset $M \subset H$, $M^\perp$ is a closed subspace of H and $M \subset (M^\perp)^\perp$.
   (b) If M is a non-closed subspace of H, show that $(M^\perp)^\perp = \overline{M}$, where the bar denotes the closure of a set. What if M is a closed subspace of H?