Inverse Problems Exercise 2
Sommersemester 2023
Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by **Tuesday 9 May, 2023, 10:15**
*Please note that there will be **no lecture on Monday 1 May** and that we will have a "bonus" **live-coding lecture** in place of the exercise session on **Tuesday 2 May, 10:15***

In tasks 1–3, let H_1 and H_2 be real Hilbert spaces. Recall that a bounded, linear operator $K: H_1 \to H_2$ is *compact* if $\overline{K(U)} \subset H_2$ is compact for all bounded sets $U \subset H_1$. Equivalently, $K: H_1 \to H_2$ is compact if and only if $(Kx_j)_{j=1}^{\infty} \subset H_2$ contains a convergent subsequence for every bounded sequence $(x_j)_{j=1}^{\infty} \subset H_1$.

1. Prove that the sum $K_1 + K_2$ of two compact operators $K_1, K_2: H_1 \to H_2$ is compact.

2. Assume that $K: H_1 \to H_2$ is a compact operator. Given bounded, linear maps $A: H \to H_1$ and $B: H_2 \to H$, where H is again a real Hilbert space, prove that KA and BK are compact.

3. Assume that $K_n: H_1 \to H_2$, $n = 1, 2, \ldots$, are compact and let $A: H_1 \to H_2$ be a bounded, linear operator such that $\|A - K_n\| \to 0$ as $n \to \infty$. Prove that A is compact.

4. Let M be a closed subspace of a real Hilbert space H and let $\lambda: M \to \mathbb{R}$ be a bounded linear functional. Show that there exists a unique linear functional $\Lambda: H \to \mathbb{R}$ such that $\Lambda(x) = \lambda(x)$ for all $x \in M$ and $\|\Lambda\| = \|\lambda\|$.