Inverse Problems Exercise 4
Sommersemester 2023
Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Tuesday 23 May, 2023, 10:15

1. Let $A \in \mathbb{R}^{m \times n}$, let $I \in \mathbb{R}^{n \times n}$ be the identity matrix, $\delta > 0$, and

$$K = \begin{bmatrix} A \\ \sqrt{\delta} I \end{bmatrix} \in \mathbb{R}^{(m+n) \times n}.$$

   Show that the singular values of K satisfy

$$\lambda_j \geq \sqrt{\delta}, \quad j = 1, \ldots, n.$$

2. Let $A \in \mathbb{R}^{m \times n}$, $y \in \mathbb{R}^m$, and let $x_\delta \in \mathbb{R}^n$ be the Tikhonov regularized solution of $Ax = y$, i.e., x_δ minimizes the functional

$$\|Ax - y\|^2 + \delta \|x\|^2, \quad \delta > 0.$$

   Let us define the "discrepancy function" $f \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f(\delta) = \|Ax_\delta - y\|^2.$$

   Prove that

$$f'(\delta) = 2\delta \langle x_\delta, (A^{\mathrm{T}} A + \delta I)^{-1} x_\delta \rangle = 2\delta x_\delta^{\mathrm{T}} (A^{\mathrm{T}} A + \delta I)^{-1} x_\delta.$$

   In particular, note that f is monotonically increasing.

3. Let us consider a simple X-ray tomography problem with limited angle data. On the course's homepage at

   `https://www.mi.fu-berlin.de/math/groups/naspde/teaching/InverseProblems.html`

   you can download the file `sino.mat`. The file contains a sparse tomography matrix $A \in \mathbb{R}^{2500 \times 1600}$, a noisy sinogram $S \in \mathbb{R}^{50 \times 50}$, and the dimension of the original object $N = 40$. In Python, you can use the script

```
import numpy as np
from scipy import sparse
import scipy.io
data = scipy.io.loadmat('sino.mat')
A = data['A']
S = data['S']
N = int(data['N'])
```

   to access the contents, while the data can be imported into MATLAB with the command

```
load sino A S N
```

Your task is to reconstruct the object corresponding to the given sinogram S. First, form the vectorized sinogram: in Python, this can be achieved with

```
y = np.transpose(S).reshape((S.size,))  # transpose required
                                        # since matrix A was
                                        # generated in MATLAB!!
```

while in MATLAB you may use `y = S(:)`. Then find a Tikhonov regularized solution, i.e., solve

$$
\begin{aligned}
x_\delta &= \arg\min_{x\in\mathbb{R}^{1600}}(\|Ax-y\|^2+\delta\|x\|^2) \\
\Leftrightarrow \quad & (A^{\mathrm{T}}A+\delta I)x_\delta = A^{\mathrm{T}}y, \quad \delta>0. \qquad (1)
\end{aligned}
$$

To find a good regularization parameter $\delta>0$, use the Morozov discrepancy principle with

$$\varepsilon = \sqrt{0.01^2\cdot 50^2} = 0.50,$$

which is the square root of the expected value for the squared norm of the (vectorized) noise, under the assumption that each pixel of the sinogram S is contaminated with normally distributed additive noise with zero mean and standard deviation 0.01. Visualize the resulting reconstruction after using `X = xdelta.reshape((40,40)).T` (Python) / `X = reshape(xdelta,40,40)` (MATLAB) to reshape the reconstruction into an image.

Remark. In Python, you can solve the system (1) somewhat efficiently using, e.g., the Scipy function

```
xdelta = sparse.linalg.lsqr(A,y,damp=np.sqrt(delta))[0]
```

(note that the `damp` parameter corresponds to $\sqrt{\delta}$, where $\delta>0$ is defined as in (1)).

In MATLAB, the equation (1) can be solved, e.g., by solving the following least squares system

```
xdelta = [A;sqrt(delta)*speye(N^2)]\[y;zeros(N^2,1)];
```