Inverse Problems Exercise 5
Sommersemester 2023
Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by **Tuesday 6 June, 2023, 10:15**
*Please note that there will be **no lecture on Monday 29 May** and that we will have a "bonus" **live-coding lecture** in place of the exercise session on **Tuesday 30 May, 10:15***

Please note that there are a total of 4 tasks in this exercise sheet.

1. Let $A \in \mathbb{R}^{m\times n}$ and $y \in \mathbb{R}^m$, and consider the equation $Ax = y$. Prove that the corresponding Landweber–Fridman iterates $\{x_k\}_{k=0}^\infty$ can be written explicitly as
$$x_k = \beta \sum_{j=0}^{k-1} (I - \beta A^{\mathrm{T}} A)^j A^{\mathrm{T}} y, \quad k = 1, 2, \ldots .$$
Moreover, with the help of a singular system (λ_j, v_j, u_j) of A, show that this is equal to
$$x_k = \sum_{j=1}^{p} \frac{1}{\lambda_j}(1 - (1 - \beta\lambda_j^2)^k)(u_j^{\mathrm{T}} y) v_j, \quad k = 0, 1, \ldots,$$
where $p = \mathrm{rank}(A)$.

2. Let $B \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix, and assume that $x \in \mathbb{R}^n$ is the solution of $Bx = w$ for some given $w \in \mathbb{R}^n$. If one approximates x using the conjugate gradient method with the initial guess $x_0 \in \mathbb{R}^n$, it is known that the k^{th} iterate satisfies (you are not required to prove this)
$$\|x - x_k\|_B \le \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^k \|x - x_0\|_B, \quad k = 1, 2, \ldots, \tag{1}$$
where $\|z\|_B^2 = z^{\mathrm{T}} B z$ and $\kappa = \mu_{\max}/\mu_{\min}$ is the condition number of B, i.e., it is the ratio of the largest eigenvalue $\mu_{\max}$ and the smallest eigenvalue $\mu_{\min}$ of B.

   (a) Show that $\mu_{\min}^{1/2}\|z\| \le \|z\|_B \le \mu_{\max}^{1/2}\|z\|$ for all $z \in \mathbb{R}^n$, where $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{R}^n$.

   (b) Using the result in part (a), derive an error estimate in the standard Euclidean norm induced by (1). That is, derive an estimate for $\|x - x_k\|$ in terms of $\|x - x_0\|$, the condition number κ, and the iteration index k.

   (c) Let $A \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$, and let $x_\delta \in \mathbb{R}^n$ be a Tikhonov regularized solution to $Ax = y$. Consider solving the corresponding normal equation
$$(A^{\mathrm{T}} A + \delta I)x = A^{\mathrm{T}} y$$
with the conjugate gradient method starting from some initial guess $x_0 \in \mathbb{R}^n$. Suppose that $\mathrm{rank}(A) < n$, which is a sound assumption (at

least up to the numerical precision) if A corresponds to an inverse/ill-posed problem. Use part (b) to write an estimate for $\|x_\delta - x_k\|$ with the help of the largest singular value of A, i.e., $\lambda_1 = \|A\|$, the regularization parameter $\delta > 0$, the iteration index k, and the initial error $\|x_\delta - x_0\|$.

3. Let us revisit the X-ray tomography problem from last week's exercises. On the course's homepage at

   `https://www.mi.fu-berlin.de/math/groups/naspde/teaching/InverseProblems.html`

   you can download the file `sino.mat`. The file contains a sparse tomography matrix $A \in \mathbb{R}^{2500\times 1600}$, a noisy sinogram $S \in \mathbb{R}^{50\times 50}$, and the dimension of the original object $N = 40$. In Python, you can use the script

```
import numpy as np
from scipy import sparse
import scipy.io
data = scipy.io.loadmat('sino.mat')
A = data['A']
S = data['S']
N = int(data['N'])
```

   to access the contents, while the data can be imported into MATLAB with the command

```
load sino A S N
```

   Your task is to reconstruct the object corresponding to the given sinogram S. First, form the vectorized sinogram: in Python, this can be achieved with

```
y = np.transpose(S).reshape((S.size,))  # transpose required
                                        # since matrix A was
                                        # generated in MATLAB!!
```

   while in MATLAB you may use `y = S(:)`. Then use Landweber–Fridman iteration to solve the equation

$$Ax = y.$$

   Use $\beta = 0.3$ and the Morozov discrepancy principle with

$$\varepsilon = \sqrt{0.01^2 \cdot 50^2} = 0.50,$$

   which is the square root of the expected value for the squared norm of the (vectorized) noise, under the assumption that each pixel of the sinogram S is contaminated with normally distributed additive noise with zero mean and standard deviation 0.01. Visualize the resulting reconstruction after using `X = xdelta.reshape((40,40)).T` (Python) / `X = reshape(xdelta,40,40)`

(MATLAB) to reshape the reconstruction into an image. Plot also the value of the residual

$$f(k) := \|Ax_k - y\|$$

as a function of k. Here, $\{x_k\}$ denote the Landweber–Fridman iterates. Why is $\beta = 10$ a bad choice? How about $\beta = 0.001$?

4. Consider solving the inverse problem in task 3 using the conjugate gradient method. Let $A \in \mathbb{R}^{2500\times 1600}$, $S \in \mathbb{R}^{50\times 50}$, and $y \in \mathbb{R}^{2500}$ be as before. Starting from the initial guess $x_0 = 0$, use the conjugate gradient method to solve the normal equation

   $$A^{\mathrm{T}}Ax = A^{\mathrm{T}}y.$$

   Use the Morozov discrepancy principle with

   $$\varepsilon = \sqrt{0.01^2 \cdot 50^2} = 0.50$$

   as the stopping rule: terminate the iteration when the norm of the residual corresponding to the *original equation* is less than ε, i.e., when

   $$\|Ax_k - y\| \leq \varepsilon.$$

   Visualize the reconstruction after using `X = xdelta.reshape((40,40)).T` (Python) or `X = reshape(xdelta,40,40)` (MATLAB) to reshape the reconstruction into an image, and plot the value of the residual

   $$f(k) := \|Ax_k - y\|$$

   as a function of k. Here, $\{x_k\}$ denote the conjugate gradient iterates. How many iterations does it take to satisfy the Morozov criterion? Visualize also the reconstruction that results from 1000 rounds of conjugate gradient iterations.