Inverse Problems Exercise 7
Sommersemester 2023
Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Tuesday 20 June, 2023, 10:15

1. Consider the following multiplicative noise model
$$y_j = a_j x_j, \quad 1 \leq j \leq n,$$
where $y, x, a \in \mathbb{R}^n$, and assume that a is a log-normally distributed multiplicative noise vector with independent components, that is, $\log a_j \sim \mathcal{N}(\log a_0, \sigma^2)$. Furthermore, a is assumed to be independent of x. By taking the logarithm, the noise model becomes additive. Using this observation, derive the likelihood density $\mathbb{P}(y|x)$ for such $x \in \mathbb{R}^n$ that $x_j > 0$ for all $j = 1, \ldots, n$.

2. Consider the linear inverse problem
$$y = Ax + \eta, \tag{1}$$
where $x \in \mathbb{R}^d$ is the unknown, $y \in \mathbb{R}^k$ is the observation, $\eta \in \mathbb{R}^k$ is additive measurement noise, and $A \in \mathbb{R}^{k \times d}$ is the matrix modeling the measurement. Moreover, suppose that the noise distribution is given by $\eta \sim \mathcal{N}(0, I)$ and the prior distribution by $x \sim \mathcal{N}(0, C)$, where $C \in \mathbb{R}^{d \times d}$ is a symmetric and positive definite matrix.

   (a) Form the posterior density $\pi^y(x)$.

   (b) Notice that the MAP estimator is precisely the minimizer of $-\log(\pi^y(x))$. Using this observation, solve the MAP estimator explicitly.

   Hint: In part (b), it may be helpful to write down the Cholesky decomposition $C^{-1} = R^{\mathrm{T}} R$, where $R \in \mathbb{R}^{d \times d}$ is an upper triangular matrix.

3. The Laplace distribution is characterized by a location parameter $t \in \mathbb{R}$ and a scale parameter $b > 0$. It has the probability density
$$\pi_{t,b}(x) = \frac{1}{2b} \exp\left(-\frac{|x - t|}{b}\right).$$
Compute the Hellinger distance
$$d_{\mathrm{H}}(\pi_{t,b}, \pi_{t,c}),$$
where $b, c > 0$.

4. Consider the linear inverse problem (1) with A, C, x, y, η, and $\pi^y(x)$ defined as in task 2. Let $A_\delta \in \mathbb{R}^{k \times d}$, $\delta > 0$, be a parametric family of matrices satisfying[†] $\|A - A_\delta\| \leq C_0 \delta$ for some constant $C_0 > 0$, and consider the *perturbed system*
$$y = A_\delta x + \eta,$$

[†]Here, $\|A\| := \sup_{x \in \mathbb{R}^d \setminus \{0\}} \frac{\|Ax\|}{\|x\|}$ denotes the matrix operator norm.

where $x \sim \mathcal{N}(0, C)$ and $\eta \sim \mathcal{N}(0, I)$ as before. Denote the posterior density of this perturbed system by $\pi^y_\delta(x)$.

Use the approximation theorem discussed during the lecture to verify that there exists $c > 0$ such that

$$d_{\mathrm{H}}(\pi^y, \pi^y_\delta) \leq c\delta$$

for sufficiently small $\delta > 0$.