Inverse Problems

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Practical matters

- **Monday May 1** (next week) is a public holiday
  $\rightarrow$ **no lecture on May 1!**
- We will have a bonus live-coding lecture on **Tuesday May 2** about Computerized Tomography in place of the usual exercise session (this material will not be essential to the course).
- The deadline for the second exercise sheet will be moved to **Tuesday May 9**. Note that tomorrow's exercise session will happen as planned.

Spectral theory of compact operators

Let E be a (complex) Banach space and $A\colon E \to E$ a bounded linear operator. The *spectrum* of operator A is denoted by

$$\sigma(A) := \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ does not have an inverse}\}.$$

Proposition

Let H be a real Hilbert space and $A\colon H \to H$ a bounded linear operator. Then

$$\sup\{|\lambda| : \lambda \in \sigma(A)\} \leq \|A\|.$$

Proof. Let $|\lambda| > \|A\|$. Then $\lambda I - A = \lambda(I - \frac{1}{\lambda}A)$, where $\|\frac{1}{\lambda}A\| < 1$. Thus $I - \frac{1}{\lambda}A$ is invertible (its inverse can be expressed as a Neumann series), and therefore the operator $\lambda I - A$ is always invertible for all $|\lambda| > \|A\|$. □

Lemma

The eigenvalues of a self-adjoint operator $A\colon H \to H$ are real-valued.

Proof. If $Ax = \lambda x$, with $x \neq 0$, then[†]

$$\lambda\langle x, x\rangle = \langle Ax, x\rangle = \langle x, A^*x\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle \quad \Rightarrow \quad \lambda = \overline{\lambda} \in \mathbb{R}$$ □

[†] If the scalar field of an inner product space is complex, then recall that the inner product needs to satisfy $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

Lemma

Let H be a real Hilbert space and let $A\colon H \to H$ be a self-adjoint operator. Then

$$\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|.$$

Proof. Let us denote $\alpha := \sup\{|\langle Ax, x\rangle| : \|x\| = 1\}$.
"$\geq$" By Cauchy–Schwarz, $|\langle Ax, x\rangle| \leq \|A\|$ for $\|x\| = 1$, and thus $\alpha \leq \|A\|$.
"$\leq$" Using $A^* = A$, we obtain the identity

$$\begin{aligned}
&\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle \\
&= \cancel{\langle Ax, x\rangle} + \langle Ax, y\rangle + \langle Ay, x\rangle + \cancel{\langle Ay, y\rangle} - \cancel{\langle Ax, x\rangle} + \langle Ax, y\rangle + \langle Ay, x\rangle - \cancel{\langle Ay, y\rangle} \\
&= 4\langle Ax, y\rangle \quad \text{for all } x, y \in H.
\end{aligned}$$

Let $x, y \in H$ be such that $\|x\| = 1 = \|y\|$. Using the inequality $|\langle Av, v\rangle| \leq \alpha\|v\|^2$ for all $v \in H$ and the parallelogram rule (exercise 1), we obtain

$$\begin{aligned}
4\langle Ax, y\rangle \leq |\langle A(x+y), x+y\rangle| + |\langle A(x-y), x-y\rangle| &\leq \alpha(\|x+y\|^2 + \|x-y\|^2) \\
&= 2\alpha(\|x\|^2 + \|y\|^2) = 4\alpha.
\end{aligned}$$

Let $\lambda = \mathrm{sign}\langle Ax, y\rangle$. Then $|\langle Ax, y\rangle| = \lambda\langle Ax, y\rangle = \langle A(\lambda x), y\rangle \leq \alpha$
$\Rightarrow \|A\| = \sup_{\|x\|=1} \sup_{\|y\|=1} |\langle Ax, y\rangle| \leq \alpha.$ ☐

If A is a compact operator, then there exists an element in H which satisfies the following.

Lemma

Let H be a real Hilbert space and let $A\colon H \to H$ be a compact, self-adjoint operator. Then

$$\|A\| = |\langle Ax_0, x_0\rangle| \quad \text{for some } x_0 \in H,\ \|x_0\| = 1. \tag{1}$$

Moreover, x_0 is an eigenvector of A, $Ax_0 = \lambda_0 x$ with $|\lambda_0| = \|A\|$.

Proof. Suppose that $A \neq 0$. By the previous lemma,

$$\|A\| = \sup\{|\langle Ax, x\rangle| : \|x\| = 1\},$$

and thus there exists a sequence $(x_n) \subset \{x \in H : \|x\| = 1\}$ such that $|\langle Ax_n, x_n\rangle| \overset{n\to\infty}{\to} \|A\|$, i.e., $\langle Ax_n, x_n\rangle \overset{n\to\infty}{\to} \lambda_0$, where $\lambda_0 \in \{-\|A\|, \|A\|\}$. Now

$$0 \leq \|Ax_n - \lambda_0 x_n\|^2 = \|Ax_n\|^2 + \lambda_0^2\|x_n\|^2 - 2\lambda_0\langle Ax_n, x_n\rangle \leq \lambda_0^2 + \lambda_0^2 - 2\lambda_0\langle Ax_n, x_n\rangle \overset{n\to\infty}{\to} 0.$$

By compactness of A, there exists a subsequence (x_{n_j}) of (x_n) and a limit $x_0 \in H$ such that $Ax_{n_j} \to x_0$. Since $Ax_{n_j} - \lambda_0 x_{n_j} \to 0$, then $\lambda x_{n_j} \to x_0$, $\|x_0\| = 1$, and $Ax_0 = \lambda_0 x_0$. □

Theorem (Spectral theorem for compact, self-adjoint operators)

Let H be a real Hilbert space and let $A\colon H \to H$ be a compact, self-adjoint operator. Then

- *each $\lambda \in \sigma(A) \setminus \{0\}$ is an eigenvalue of A;*
- *0 is the only limit point of $\sigma(A)$, i.e., if there are an infinite number of eigenvalues $(\lambda_n) \subset \mathbb{R}$, then $\lim_n \lambda_n = 0$;*
- *the eigenvectors $(u_n) \subset H$ form an orthonormal sequence such that*

$$Ax = \sum_n \lambda_n \langle x, u_n \rangle u_n.$$

Proof. We have already established that there exists $u_0 \in H$ s.t. $Au_0 = \lambda_0 u_0$, $|\lambda_0| = \|A\|$ and $\|u_0\| = 1$. Define $H_1 := \{u_0\}^{\perp}$. If $y \in H_1$, then

$$\langle Ay, u_0 \rangle = \langle y, Au_0 \rangle = \lambda_1 \langle y, u_0 \rangle = 0,$$

which means that $A|_{H_1} \colon H_1 \to H_1$ is a compact, self-adjoint operator.

By (1), there exists $u_1 \in H_1$ such that

$$\|A|_{H_1}\| = |\langle u_1, Au_1 \rangle|$$

with $Au_1 = \lambda_1 u_1$, where $|\lambda_1| \leq |\lambda_0|$ and $\langle u_0, u_1 \rangle = 0$.

Next, let $H_2 := \{u_0, u_1\}^\perp$. As before, $A|_{H_2} \colon H_2 \to H_2$ is a compact, self-adjoint operator and (1) again implies that there exists $u_2 \in H_2$ such that $Au_2 = \lambda_2 u_2$, where $|\lambda_2| \leq |\lambda_1| \leq |\lambda_0|$ and $\|u_2\| = 1$.

Proceeding inductively, we obtain $H_n := \{u_0, \ldots, u_{n-1}\}^\perp \subset H_{n-1}$, where $A|_{H_n} \colon H_n \to H_n$ is compact and self-adjoint, $|\lambda_n| = \|A|_{H_n}\|$, $|\lambda_n| \leq |\lambda_{n-1}| \leq \cdots \leq |\lambda_0|$ and $Au_n = \lambda_n u_n$ for some $u_n \in H_n$, $\|u_n\| = 1$.

If $\dim \mathrm{Ran}(A) = \infty$, we claim that $|\lambda_n| \to 0$ as $n \to \infty$. Since $u_k \perp u_j$ whenever $j \neq k$, we deduce that

$$|\lambda_j|^2 + |\lambda_k|^2 = \|\lambda_k u_k - \lambda_j u_j\|^2 = \|Au_k - Au_j\|^2.$$

Note that (λ_j^2) is convergent as a bounded, monotonic sequence. Since (u_j) is bounded and A is compact, (Au_j) contains a convergent subsequence – and hence it contains a Cauchy subsequence. This implies that (λ_j^2) contains a subsequence which converges to 0. Since (λ_j^2) is a convergent sequence, it follows that $\lim_{j\to\infty} \lambda_j = 0$.

Let $M := \operatorname{span}\{u_n \mid n \in \mathbb{N}\}^{\perp}$. The previous discussion implies that $A|_M = 0$. Let $H_\infty := \overline{\operatorname{span}\{u_n \mid u_n \in \mathbb{N}\}}$. By the orthogonal decomposition $H = M \oplus H_\infty$, the orthogonal projection $P \colon H \to H_\infty$ can be written as

$$Px = \sum_n \langle x, u_n \rangle u_n, \quad x \in H \qquad \text{(proof left as an exercise)}$$

and therefore

$$Ax = APx = A\Big(\sum_n \langle x, u_n \rangle u_n\Big) = \sum_n \langle x, u_n \rangle A u_n = \sum_n \lambda_n \langle x, u_n \rangle u_n,$$

as desired.

Finally, to see that each $\lambda \in \sigma(A) \setminus \{0\}$ is an eigenvalue, suppose that $\lambda \notin \overline{\{\lambda_n \mid n \in \mathbb{N}\}} \cup \{0\}$. Then there exists $\delta > 0$ such that $|\lambda - \lambda_n| > \delta$ for all $n \in \mathbb{N}$ and $|\lambda| > \delta$. If $Q \colon H \to M$ is an orthogonal projection, then

$$(\lambda I - A)^{-1} x = \sum_n \frac{1}{\lambda - \lambda_n} \langle x, u_n \rangle u_n + \frac{1}{\lambda} Q x, \quad x \in H,$$

is bounded by the previous discussion, i.e., $\lambda \notin \sigma(A)$. $\square$

Our goal is to obtain a spectral expansion for all compact operators $A\colon H_1 \to H_2$. To begin with, note that if $A\colon H_1 \to H_2$ is a compact operator, then $A^*A\colon H_1 \to H_1$ is compact and self-adjoint since

$$\langle A^*Ax, y\rangle_{H_1} = \langle Ax, Ay\rangle_{H_2} = \langle x, A^*Ay\rangle_{H_1} \quad \text{for all} \;\; x, y \in H_1.$$

Note in addition that the eigenvalues of A^*A are nonnegative: if $A^*Av_n = \lambda_n v_n$, $\|v_n\|_{H_1} = 1$, then

$$\lambda_n = \lambda_n \|v_n\|_{H_1}^2 = \langle A^*Av_n, v_n\rangle_{H_1} = \|Av_n\|_{H_2}^2 \geq 0.$$

In particular, we can write down the eigendecomposition

$$A^*Ax = \sum_n \lambda_n \langle x, v_n\rangle_{H_1} v_n,$$

where $(v_n) \subset H_1$ is an orthonormal sequence of eigenvectors.

Lemma

Let H_1 and H_2 be real Hilbert spaces and let $A\colon H_1 \to H_2$ be a compact operator. Then there exist orthonormal sequences $(v_n) \subset H_1$ and $(w_n) \subset H_2$ such that

$$Av_n = \sqrt{\lambda_n} w_n \quad \text{and} \quad A^* w_n = \sqrt{\lambda_n} v_n, \tag{2}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ are the nonzero eigenvalues of A^*A. Define $|A|\colon H_1 \to H_2$ by setting $|A|x = \sum_n \sqrt{\lambda_n} \langle x, v_n \rangle_{H_1} w_n$. Then

$|A|$ is compact and $|A|^*|A| = A^*A$.

Proof. Let $(v_n) \subset H_1$ denote the orthonormal sequence of eigenfunctions of A^*A, i.e.,

$$A^*Av_n = \lambda_n v_n$$

and define a second sequence by

$$w_n = \frac{1}{\sqrt{\lambda_n}} Av_n.$$

Straightforward computations show that (2) holds as well as $\langle w_n, w_n \rangle_{H_2} = 1$ and $\langle w_n, w_m \rangle_{H_2} = 0$ whenever $n \neq m$.

Next, let us show that $|A|: H_1 \to H_2$ is compact. It follows from the generalized Pythagorean theorem and Bessel's inequality that

$$\Big\| |A|x - \sum_{n=1}^{m} \sqrt{\lambda_n} \langle x, v_n \rangle_{H_1} w_n \Big\|_{H_2}^2 = \Big\| \sum_{n=m+1}^{\infty} \sqrt{\lambda_n} \langle x, v_n \rangle_{H_1} w_n \Big\|_{H_2}^2$$

$$= \sum_{n=m+1}^{\infty} |\lambda_n| |\langle x, v_n \rangle_{H_1}|^2 \leq \sup_{n \geq m+1} |\lambda_n| \cdot \|x\|^2$$

$$\leq \sup_{n \geq m+1} |\lambda_n| \quad \text{for all } \|x\|_{H_1} \leq 1.$$

Thus $\| |A| - \sum_{n=1}^{m} \sqrt{\lambda_n} \langle \cdot, v_n \rangle_{H^1} w_n \| \leq \sup_{n \geq m+1} \sqrt{\lambda_n} \to 0$ as $m \to \infty$. Since the operators $x \mapsto \langle x, v_n \rangle_{H_1} w_n$ have 1-dimensional range, they are compact. Moreover, finite sums $\sum_{n=1}^{m} \sqrt{\lambda_n} \langle \cdot, v_n \rangle_{H_1} w_n$ of compact operators are compact and, in consequence, their limiting operator $|A|$ is compact. (See, e.g., properties of compact operators from the lecture notes of week 1.)

Finally, we wish to show that $|A|^*|A| = A^*A$. It is not difficult to check that

$$|A|^* = \sum_n \sqrt{\lambda_n} \langle \cdot, w_n \rangle_{H_2} v_n.$$

Let $x \in H_1$. A direct computation then reveals that

$$\begin{aligned} |A|^*|A|x &= |A|^* \left(\sum_n \sqrt{\lambda_n} \langle x, v_n \rangle_{H_1} w_n \right) \\ &= \sum_m \sqrt{\lambda_m} \left\langle \sum_n \sqrt{\lambda_n} \langle x, v_n \rangle_{H_1} w_n, w_m \right\rangle_{H_2} v_m \\ &= \sum_{m,n} \sqrt{\lambda_m \lambda_n} \langle x, v_n \rangle_{H_1} \langle w_n, w_m \rangle_{H_2} v_m \\ &= \sum_n \lambda_n \langle x, v_n \rangle_{H_1} v_n = A^*Ax, \end{aligned}$$

where we used $\langle w_n, w_n \rangle_{H_2} = 1$ and $\langle w_n, w_m \rangle_{H_2} = 0$ whenever $n \neq m$. □

Proposition (Polar decomposition)

Let H_1 and H_2 be real Hilbert spaces, $A\colon H_1 \to H_2$ a compact operator, and let $|A|$ be defined as before. Then there exists a bounded, linear operator $U\colon H_2 \to H_2$ such that

$$A = U|A|,$$

where $\|Ux\|_{H_2} = \|x\|_{H_2}$ for all $x \in \overline{\mathrm{Ran}(|A|)}$ and $Uy = 0$ for all $y \in \mathrm{Ran}(|A|)^{\perp}$.

Proof. If $x \in H_1$, then

$$\||A|x\|_{H_2}^2 = \langle |A|x, |A|x\rangle_{H_2} = \langle x, |A|^*|A|x\rangle_{H_1} = \langle x, A^*Ax\rangle_{H_1} = \langle Ax, Ax\rangle_{H_2} = \|Ax\|_{H_2}^2.$$

We can define a linear mapping $U\colon \mathrm{Ran}(|A|) \to \mathrm{Ran}(A)$ by setting $U(|A|x) = Ax$ for $x \in H_1$. Since the above formula implies

$$\|U(|A|x)\| = \|Ax\| = \||A|x\| \quad \text{for all } |A|x \in \mathrm{Ran}(|A|),$$

there exists a unique extension $U\colon \overline{\mathrm{Ran}(|A|)} \to \overline{\mathrm{Ran}(A)}$ s.t. $\|Ux\| = \|x\|$ for all $x \in \overline{\mathrm{Ran}(|A|)}$. Finally, since we have the orthogonal decomposition $H_2 = \overline{\mathrm{Ran}(|A|)} \oplus \mathrm{Ran}(|A|)^{\perp}$, we can set $Uy = 0$ for all $y \in \mathrm{Ran}(|A|)^{\perp}$. □

Theorem (Singular value decomposition of compact operators)

Let H_1 and H_2 be real Hilbert spaces and let $A\colon H_1 \to H_2$ be a compact operator. Then there exists a (possibly finite) sequence of positive real numbers $(\lambda_n) \subset \mathbb{R}$ with $\lim_{n\to\infty} \lambda_n = 0$ and (possibly finite) orthonormal sequences $(v_n) \subset H_1$ and $(u_n) \subset H_2$ such that

$$Ax = \sum_n \lambda_n \langle x, v_n \rangle_{H_1} u_n, \quad x \in H_1.$$

Proof. The operator $A^*A\colon H_1 \to H_1$ is compact and self-adjoint. Let

$$A^*Ax = \sum_n \lambda_n \langle x, v_n \rangle_{H_1} v_n, \quad x \in H_1,$$

be its eigendecomposition. Moreover, let

$$|A|x = \sum_n \sqrt{\lambda_n} \langle x, v_n \rangle_{H_1} w_n, \quad x \in H_1,$$

be defined as before. Then using the polar decomposition:

$$Ax = U|A|x = U\left(\sum_n \sqrt{\lambda_n} \langle x, v_n \rangle_{H_1} w_n \right) = \sum_n \sqrt{\lambda_n} \langle x, v_n \rangle_{H_1} \underbrace{U(w_n)}_{=: u_n}.$$

Since U is an isometry in $\overline{\mathrm{Ran}(|A|)}$, $\langle w_n, w_m \rangle_{H_2} = \langle U(w_n), U(w_m) \rangle_{H_2}$ (exercise 1). Therefore (u_n) is also an orthonormal sequence. $\square$