Inverse Problems

Sommersemester 2023

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Numerical example: X-ray tomography

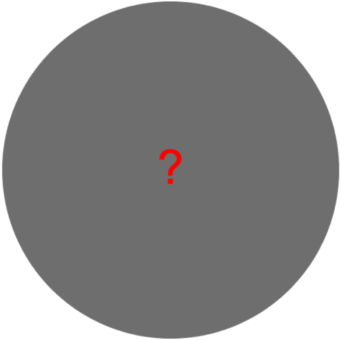
As an application, we consider X-ray tomography and describe here the construction of the tomography matrix. We will return to this example on Tuesday May 30 when we will discuss total variation regularization for X-ray tomography.

The following content follows roughly the material presented in the following monographs.

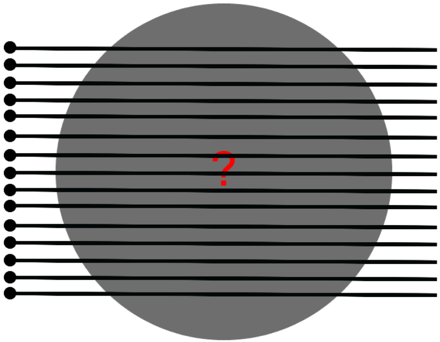
- J. Kaipio and E. Somersalo. Statistical and Computational Inverse Problems. 2005.
- J. L. Mueller and S. Siltanen. Linear and Nonlinear Inverse Problems with Practical Applications. 2012.

ASTRA Toolbox for 2D and 3D tomography:
https://www.astra-toolbox.com/

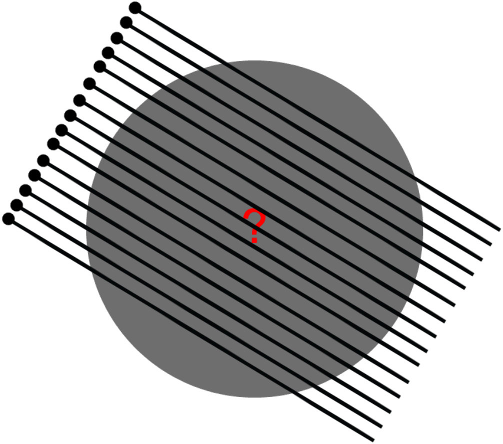
Parallel-beam X-ray tomography



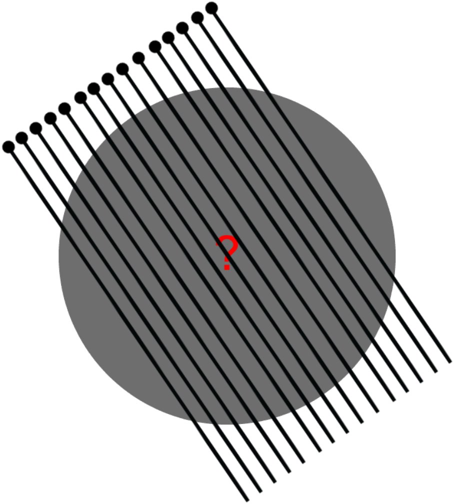
Parallel-beam X-ray tomography



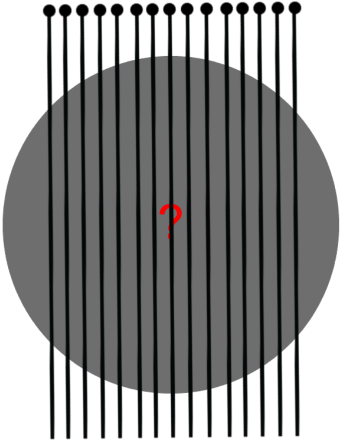
Parallel-beam X-ray tomography



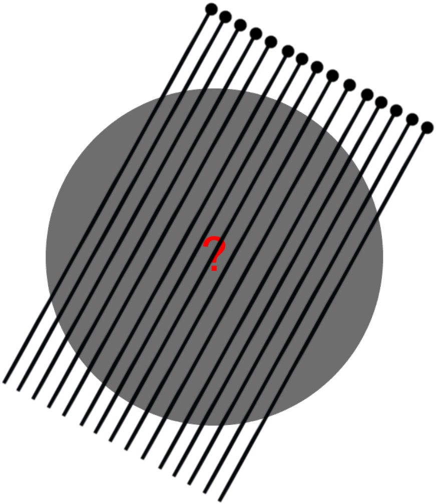
Parallel-beam X-ray tomography



Parallel-beam X-ray tomography



Parallel-beam X-ray tomography



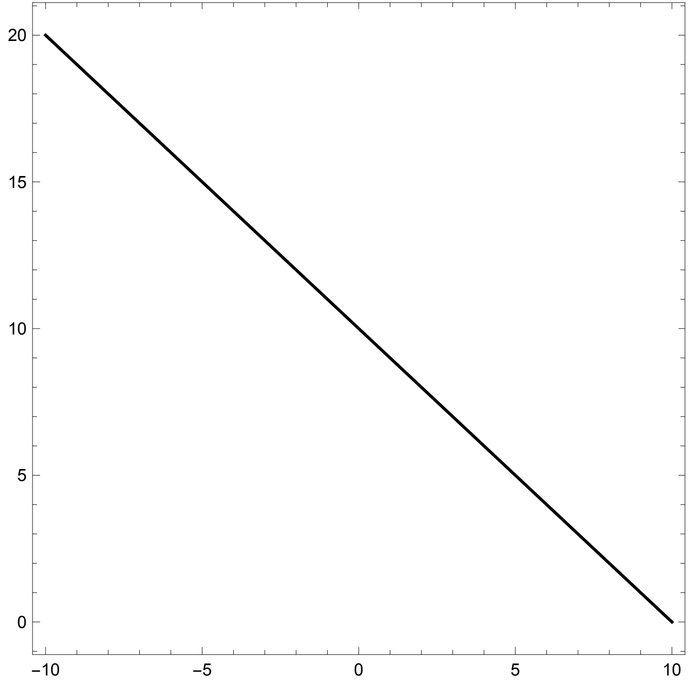
Radon transform in $\mathbb{R}^2$

Let L be a straight line in $\mathbb{R}^2$.

Any line in $\mathbb{R}^2$ can be parameterized as

$$L = \{s\omega + t\omega^{\perp};\ t \in \mathbb{R}\} \quad \text{for some } s \in \mathbb{R} \text{ and } \omega \in S^1,$$

where $\omega^{\perp} \perp \omega$.


20
15
10
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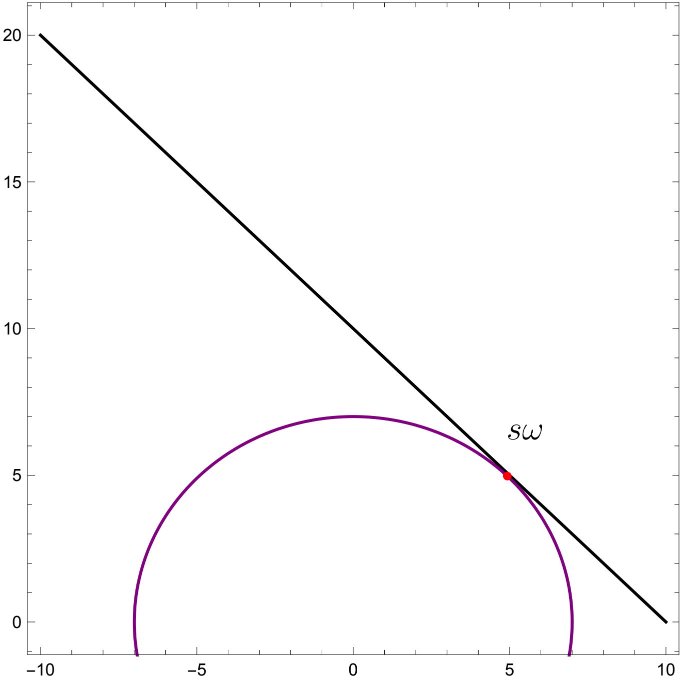
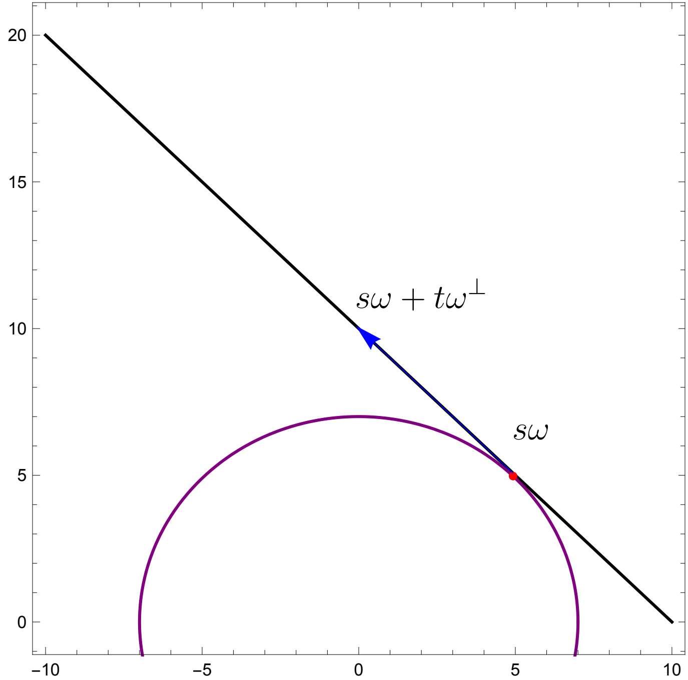


$s\omega$
20
15
10
5
0
−10
−5
0
5
10



$s\omega + t\omega^{\perp}$
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20
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where $\omega^{\perp} \perp \omega$.

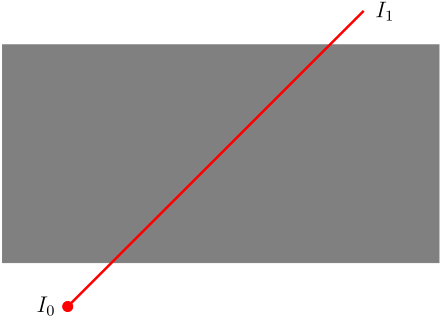
Writing $\omega = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, we get

$$L = L(s,\theta) = \left\{ s \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + t \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix};\ t \in \mathbb{R} \right\}, \quad s \in \mathbb{R} \text{ and } \theta \in [0,\pi).$$

The *Radon transform* of a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ on L is defined as

$$\mathcal{R}f(L) = \int_L f(\boldsymbol{x})\,|\mathrm{d}\boldsymbol{x}| = \int_{-\infty}^{\infty} f(s\cos\theta + t\sin\theta, s\sin\theta - t\cos\theta)\,\mathrm{d}t.$$

Let f be a nonnegative function modeling X-ray attenuation (density) inside a physical body.



Beer–Lambert law:

$$\mathcal{R}f(L) = \log \frac{I_0}{I_1}.$$

$f_{9,0}$	$f_{9,1}$	$f_{9,2}$	$f_{9,3}$	$f_{9,4}$	$f_{9,5}$	$f_{9,6}$	$f_{9,7}$	$f_{9,8}$	$f_{9,9}$
$f_{8,0}$	$f_{8,1}$	$f_{8,2}$	$f_{8,3}$	$f_{8,4}$	$f_{8,5}$	$f_{8,6}$	$f_{8,7}$	$f_{8,8}$	$f_{8,9}$
$f_{7,0}$	$f_{7,1}$	$f_{7,2}$	$f_{7,3}$	$f_{7,4}$	$f_{7,5}$	$f_{7,6}$	$f_{7,7}$	$f_{7,8}$	$f_{7,9}$
$f_{6,0}$	$f_{6,1}$	$f_{6,2}$	$f_{6,3}$	$f_{6,4}$	$f_{6,5}$	$f_{6,6}$	$f_{6,7}$	$f_{6,8}$	$f_{6,9}$
$f_{5,0}$	$f_{5,1}$	$f_{5,2}$	$f_{5,3}$	$f_{5,4}$	$f_{5,5}$	$f_{5,6}$	$f_{5,7}$	$f_{5,8}$	$f_{5,9}$
$f_{4,0}$	$f_{4,1}$	$f_{4,2}$	$f_{4,3}$	$f_{4,4}$	$f_{4,5}$	$f_{4,6}$	$f_{4,7}$	$f_{4,8}$	$f_{4,9}$
$f_{3,0}$	$f_{3,1}$	$f_{3,2}$	$f_{3,3}$	$f_{3,4}$	$f_{3,5}$	$f_{3,6}$	$f_{3,7}$	$f_{3,8}$	$f_{3,9}$
$f_{2,0}$	$f_{2,1}$	$f_{2,2}$	$f_{2,3}$	$f_{2,4}$	$f_{2,5}$	$f_{2,6}$	$f_{2,7}$	$f_{2,8}$	$f_{2,9}$
$f_{1,0}$	$f_{1,1}$	$f_{1,2}$	$f_{1,3}$	$f_{1,4}$	$f_{1,5}$	$f_{1,6}$	$f_{1,7}$	$f_{1,8}$	$f_{1,9}$
$f_{0,0}$	$f_{0,1}$	$f_{0,2}$	$f_{0,3}$	$f_{0,4}$	$f_{0,5}$	$f_{0,6}$	$f_{0,7}$	$f_{0,8}$	$f_{0,9}$

Let us consider the computational domain $[-1,1]^2$. We divide this region into $n \times n$ pixels and approximate the density by a piecewise constant function with constant value

$$f_{i,j} \text{ in pixel } P_{i,j}$$

for $i, j \in \{0, \ldots, n-1\}$.

$$P_{i,j} := \{(x,y);\ -1 + 2\,\tfrac{j}{n} < x < -1 + 2\,\tfrac{j+1}{n},\ -1 + 2\,\tfrac{i}{n} < y < -1 + 2\,\tfrac{i+1}{n}\}$$

x_{90}	x_{91}	x_{92}	x_{93}	x_{94}	x_{95}	x_{96}	x_{97}	x_{98}	x_{99}
x_{80}	x_{81}	x_{82}	x_{83}	x_{84}	x_{85}	x_{86}	x_{87}	x_{88}	x_{89}
x_{70}	x_{71}	x_{72}	x_{73}	x_{74}	x_{75}	x_{76}	x_{77}	x_{78}	x_{79}
x_{60}	x_{61}	x_{62}	x_{63}	x_{64}	x_{65}	x_{66}	x_{67}	x_{68}	x_{69}
x_{50}	x_{51}	x_{52}	x_{53}	x_{54}	x_{55}	x_{56}	x_{57}	x_{58}	x_{59}
x_{40}	x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	x_{46}	x_{47}	x_{48}	x_{49}
x_{30}	x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	x_{36}	x_{37}	x_{38}	x_{39}
x_{20}	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}	x_{28}	x_{29}
x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}
x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9

It is convenient to reshape the matrix/image $(f_{i,j})$ into a vector x of length n^2 so that

$$x_{in+j} = f_{i,j}, \quad i, j \in \{0, \ldots, n-1\}.$$

The image on the left illustrates the new numbering corresponding to the pixels.

Note that `x = f.reshape((n*n,1))` and `f = x.reshape((n,n))`.
(In MATLAB: `x = f(:)` and `f = reshape(x,n,n)`).

Measurement model

Let us consider a measurement setup where we take X-ray measurements of an object using K X-rays $L(s_0,\theta),\ldots,L(s_{K-1},\theta)$ taken at angles $\theta\in\{\theta_0,\ldots,\theta_{M-1}\}$. The total number of X-rays is $Q = MK$.

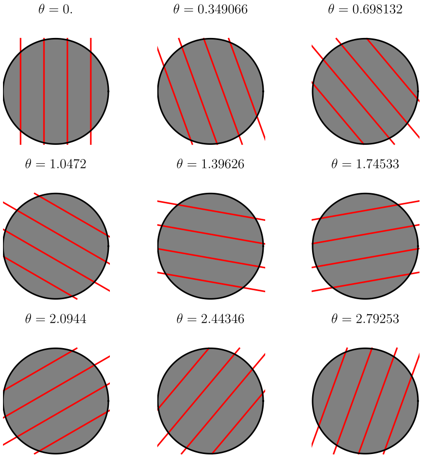
For brevity, let us write $L_{mK+k} := L(s_k,\theta_m)$ for $k\in\{0,\ldots,K-1\}$ and $m\in\{0,\ldots,M-1\}$.

The measurement model is

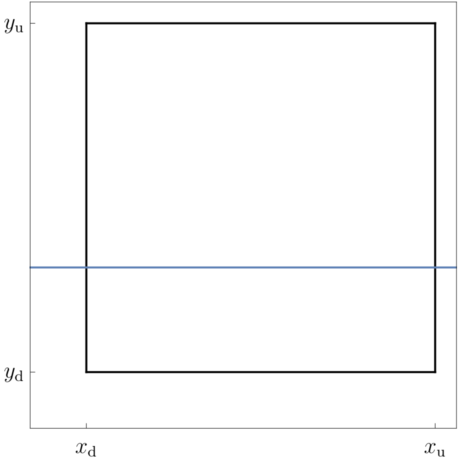
$$y = \begin{bmatrix} \int_{L_0} f(\boldsymbol{x})|\mathrm{d}\boldsymbol{x}| \\ \vdots \\ \int_{L_{Q-1}} f(\boldsymbol{x})|\mathrm{d}\boldsymbol{x}| \end{bmatrix} + \eta \approx \begin{bmatrix} \sum_{j=0}^{n^2-1} A_{0,j}x_j \\ \vdots \\ \sum_{j=0}^{n^2-1} A_{Q-1,j}x_j \end{bmatrix} + \eta = Ax + \eta,$$

where $A\in\mathbb{R}^{Q\times n^2}$ and $A_{i,j}$ is the distance that ray L_i travels through pixel j. Here, x is a vector containing the (piecewise constant) densities within each pixel and η is measurement noise.

$$L_{mK+k} = \left\{ s_k \begin{bmatrix} \cos\theta_m \\ \sin\theta_m \end{bmatrix} + t \begin{bmatrix} \sin\theta_m \\ -\cos\theta_m \end{bmatrix} ;\ t \in \mathbb{R} \right\}, \quad \begin{matrix} k = 0, \ldots, K-1, \\ m = 0, \ldots, M-1. \end{matrix}$$



Pixel-by-pixel construction of the tomography matrix A
(See the files `tomodemo.py`/`tomodemo.m` on the course page!)



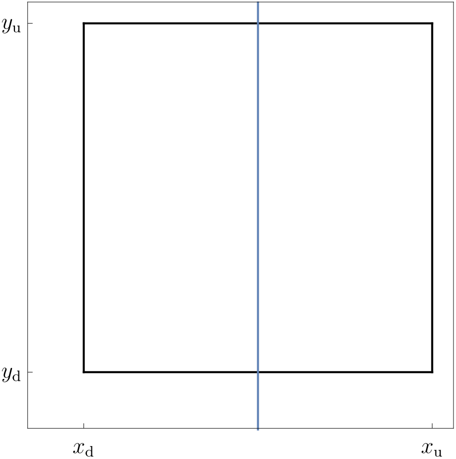
Case $\cos\theta = 0$ and $\sin\theta = 1$:

$$\begin{bmatrix} x_d \\ y_d \end{bmatrix} \le \begin{bmatrix} s\cos\theta + t\sin\theta \\ s\sin\theta - t\cos\theta \end{bmatrix} < \begin{bmatrix} x_u \\ y_u \end{bmatrix}$$
$$\Leftrightarrow \begin{bmatrix} x_d \\ y_d \end{bmatrix} \le \begin{bmatrix} t \\ s \end{bmatrix} < \begin{bmatrix} x_u \\ y_u \end{bmatrix}.$$

The distance that ray L_m travels through pixel k is

$$A_{m,k} = \int_{L_m} \chi_k \, |\mathrm{d}\boldsymbol{x}| = \int_{\substack{x_d \le t < x_u \\ y_d \le s < y_u}} \mathrm{d}t = \begin{cases} x_u - x_d & \text{if } y_d \le s < y_u, \\ 0 & \text{otherwise.} \end{cases}$$

N.B. In here and in the following, $\chi_k = \chi_k(\boldsymbol{x})$ denotes the characteristic function of the k^{th} pixel. In the above illustration, the pixel is denoted by the rectangle $[x_d, x_u) \times [y_d, y_u)$.



Case $\cos\theta = 1$ and $\sin\theta = 0$:

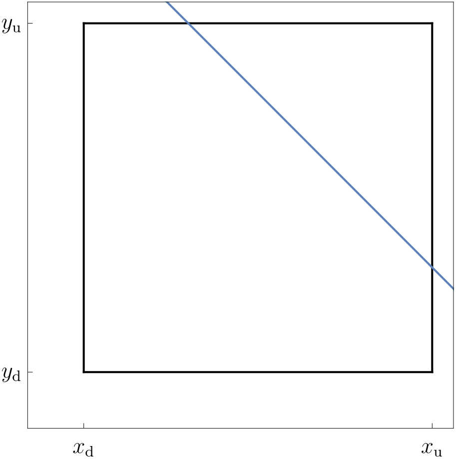
$$\begin{bmatrix} x_d \\ y_d \end{bmatrix} \leq \begin{bmatrix} s\cos\theta + t\sin\theta \\ s\sin\theta - t\cos\theta \end{bmatrix} < \begin{bmatrix} x_u \\ y_u \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} x_d \\ y_d \end{bmatrix} \leq \begin{bmatrix} s \\ -t \end{bmatrix} < \begin{bmatrix} x_u \\ y_u \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} x_d \\ -y_u \end{bmatrix} < \begin{bmatrix} s \\ t \end{bmatrix} \leq \begin{bmatrix} x_u \\ -y_d \end{bmatrix}.$$

The distance that ray L_m travels through pixel k is

$$A_{m,k} = \int_{L_m} \chi_k \, |\mathrm{d}\boldsymbol{x}| = \int\limits_{\substack{-y_u < t \leq -y_d \\ x_d < s \leq x_u}} \mathrm{d}t = \begin{cases} y_u - y_d & \text{if } x_d < s \leq x_u, \\ 0 & \text{otherwise.} \end{cases}$$



Case $\cos\theta > 0$:

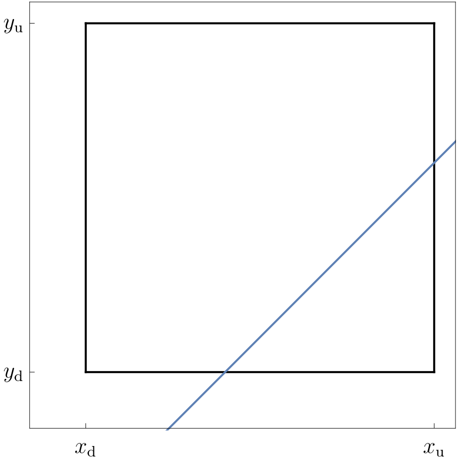
$$\begin{bmatrix} x_d \\ y_d \end{bmatrix} < \begin{bmatrix} s\cos\theta + t\sin\theta \\ s\sin\theta - t\cos\theta \end{bmatrix} < \begin{bmatrix} x_u \\ y_u \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \frac{x_d - s\cos\theta}{\sin\theta} \\ \frac{s\sin\theta - y_u}{\cos\theta} \end{bmatrix} < \begin{bmatrix} t \\ t \end{bmatrix} < \begin{bmatrix} \frac{x_u - s\cos\theta}{\sin\theta} \\ \frac{s\sin\theta - y_d}{\cos\theta} \end{bmatrix}.$$

The distance that ray L_m travels through pixel k is

$$A_{m,k} = \int_{L_m} \chi_k \, |\mathrm{d}\boldsymbol{x}| = \int_{\max\left\{\frac{x_d - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_u}{\cos\theta}\right\} < t < \min\left\{\frac{x_u - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_d}{\cos\theta}\right\}} \mathrm{d}t$$

$$= \left(\min\left\{ \frac{x_u - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_d}{\cos\theta} \right\} - \max\left\{ \frac{x_d - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_u}{\cos\theta} \right\} \right)_+ .$$



Case $\cos\theta < 0$:

$$\begin{bmatrix} x_\mathrm{d} \\ y_\mathrm{d} \end{bmatrix} < \begin{bmatrix} s\cos\theta + t\sin\theta \\ s\sin\theta - t\cos\theta \end{bmatrix} < \begin{bmatrix} x_\mathrm{u} \\ y_\mathrm{u} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \frac{x_\mathrm{d} - s\cos\theta}{\sin\theta} \\ s\sin\theta - y_\mathrm{u} \end{bmatrix} < \begin{bmatrix} t \\ t\cos\theta \end{bmatrix} < \begin{bmatrix} \frac{x_\mathrm{u} - s\cos\theta}{\sin\theta} \\ s\sin\theta - y_\mathrm{d} \end{bmatrix}$$

$$\overset{!}{\Leftrightarrow} \begin{bmatrix} \frac{x_\mathrm{d} - s\cos\theta}{\sin\theta} \\ \frac{s\sin\theta - y_\mathrm{d}}{\cos\theta} \end{bmatrix} < \begin{bmatrix} t \\ t \end{bmatrix} < \begin{bmatrix} \frac{x_\mathrm{u} - s\cos\theta}{\sin\theta} \\ \frac{s\sin\theta - y_\mathrm{u}}{\cos\theta} \end{bmatrix}.$$

The distance that ray L_m travels through pixel k is

$$A_{m,k} = \int_{L_m} \chi_k \, |\mathrm{d}\boldsymbol{x}| = \int_{\max\left\{\frac{x_\mathrm{d} - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_\mathrm{d}}{\cos\theta}\right\} < t < \min\left\{\frac{x_\mathrm{u} - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_\mathrm{u}}{\cos\theta}\right\}} \mathrm{d}t$$

$$= \left(\min\left\{\frac{x_\mathrm{u} - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_\mathrm{u}}{\cos\theta}\right\} - \max\left\{\frac{x_\mathrm{d} - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_\mathrm{d}}{\cos\theta}\right\}\right)_+.$$

Discussion

Tomography problems can be classified into three classes based on the nature of the measurement data:

- Full angle tomography
  - Sufficient number of measurements from all angles $\rightarrow$ not a very ill-posed problem.
- Limited angle tomography
  - Data collected from a restricted angle of view $\rightarrow$ reconstructions very sensitive to measurement error and it is not possible to reconstruct the object perfectly (even with noiseless data). Applications include, e.g., dental imaging.
- Sparse data tomography
  - The data consist of only a few projection images, possibly from any direction $\rightarrow$ extremely ill-posed inverse problem and prior knowledge necessary for successful reconstructions. (E.g., minimizing a patient's radiation dose.)