Domain uncertainty quantification using periodic random variables with application to elliptic PDEs

H. Hakula[1], H. Harbrecht[2], V. Kaarnioja[3], F. Y. Kuo[4], and I. H. Sloan[4]

[1] *Aalto University, Department of Mathematics and Systems Analysis, P.O. Box 11100, 00076 Aalto, Finland*

[2] *University of Basel, Department of Mathematics and Computer Science, Spiegelgasse 1, 4051 Basel, Switzerland*

[3] *Free University of Berlin, Department of Mathematics and Computer Science, Arnimallee 6, 14195 Berlin, Germany*

[4] *University of New South Wales, School of Mathematics and Statistics, Sydney NSW 2052, Australia*

Problem setting

Let $(\Omega, \Gamma, \mathbb{P})$ be a probability space. We consider the Poisson problem

$$-\Delta u(\boldsymbol{x}, \omega) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in D(\omega),$$
$$u(\boldsymbol{x}, \omega) = 0, \quad \boldsymbol{x} \in \partial D(\omega),$$

subject to an *uncertain domain* $D(\omega) \subset \mathbb{R}^d$, $d \in \{2,3\}$, for almost every $\omega \in \Omega$.

Domain mapping method: Let $D_{\rm ref} \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a fixed *reference domain*. Define perturbation field $\boldsymbol{V}(\cdot, \omega) \colon \overline{D_{\rm ref}} \to \mathbb{R}^d$, which we assume is given explicitly.

Domain parameterization

Let $U := [-\frac{1}{2}, \frac{1}{2}]^{\mathbb{N}}$ and let $\boldsymbol{V} \colon \overline{D_{\rm ref}} \times U \to \mathbb{R}^d$ be a vector field such that, for $\boldsymbol{x} \in D_{\rm ref}$ and $\boldsymbol{y} \in U$,

$$\boldsymbol{V}(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{x} + \frac{1}{\sqrt{6}} \sum_{i=1}^{\infty} \sin(2\pi y_i) \boldsymbol{\psi}_i(\boldsymbol{x}),$$

with *stochastic fluctuations* $\boldsymbol{\psi}_i \colon D_{\rm ref} \to \mathbb{R}^d$. Denoting the Jacobian matrix of $\boldsymbol{\psi}_i$ by $\boldsymbol{\psi}_i'$, the Jacobian matrix $J(\cdot, \boldsymbol{y}) \colon D_{\rm ref} \to \mathbb{R}^{d \times d}$ of vector field $\boldsymbol{V}(\cdot, \boldsymbol{y})$ is, for $\boldsymbol{x} \in D_{\rm ref}$ and $\boldsymbol{y} \in U$,

$$J(\boldsymbol{x}, \boldsymbol{y}) := I + \frac{1}{\sqrt{6}} \sum_{i=1}^{\infty} \sin(2\pi y_i) \boldsymbol{\psi}_i'(\boldsymbol{x}).$$

The family of *admissible domains* $\{D(\boldsymbol{y})\}_{\boldsymbol{y} \in U}$ is parameterized for all $\boldsymbol{y} \in U$ by

$$D(\boldsymbol{y}) := \boldsymbol{V}(D_{\rm ref}, \boldsymbol{y}),$$

and the *hold-all domain* is defined by setting

$$\mathcal{D} := \bigcup_{\boldsymbol{y} \in U} D(\boldsymbol{y}).$$

Standing assumptions

We make the following *standing assumptions*:

(A1) For each $\boldsymbol{y} \in U$, $\boldsymbol{V}(\cdot, \boldsymbol{y}) : \overline{D_{\rm ref}} \to \mathbb{R}^d$ is an invertible, twice continuously differentiable vector field.

(A2) For some $C > 0$, there holds

$$\|\boldsymbol{V}(\cdot, \boldsymbol{y})\|_{\mathcal{C}^2(\overline{D_{\rm ref}})} \leq C,$$
$$\|\boldsymbol{V}^{-1}(\cdot, \boldsymbol{y})\|_{\mathcal{C}^2(\overline{D(\boldsymbol{y})})} \leq C$$

for all $\boldsymbol{y} \in U$.

(A3) There exist constants $0 < \underline{\sigma} \leq 1 \leq \overline{\sigma} < \infty$ such that

$$\underline{\sigma} \leq \min \sigma(J(\boldsymbol{x}, \boldsymbol{y})) \leq \max \sigma(J(\boldsymbol{x}, \boldsymbol{y})) \leq \overline{\sigma}$$

for all $\boldsymbol{x} \in D_{\rm ref}$ and $\boldsymbol{y} \in U$, where $\sigma(J(\boldsymbol{x}, \boldsymbol{y}))$ denotes the set of all singular values of matrix $J(\boldsymbol{x}, \boldsymbol{y})$.

(A4) There holds $\|\boldsymbol{\psi}_i\|_{W^{1,\infty}(D_{\rm ref};\mathbb{R}^d)} < \infty$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \|\boldsymbol{\psi}_i\|_{W^{1,\infty}(D_{\rm ref};\mathbb{R}^d)} < \infty$.

(A5) For some $p \in (0,1)$, there holds

$$\sum_{i=1}^{\infty} \|\boldsymbol{\psi}_i\|_{W^{1,\infty}(D_{\rm ref};\mathbb{R}^d)}^p < \infty.$$

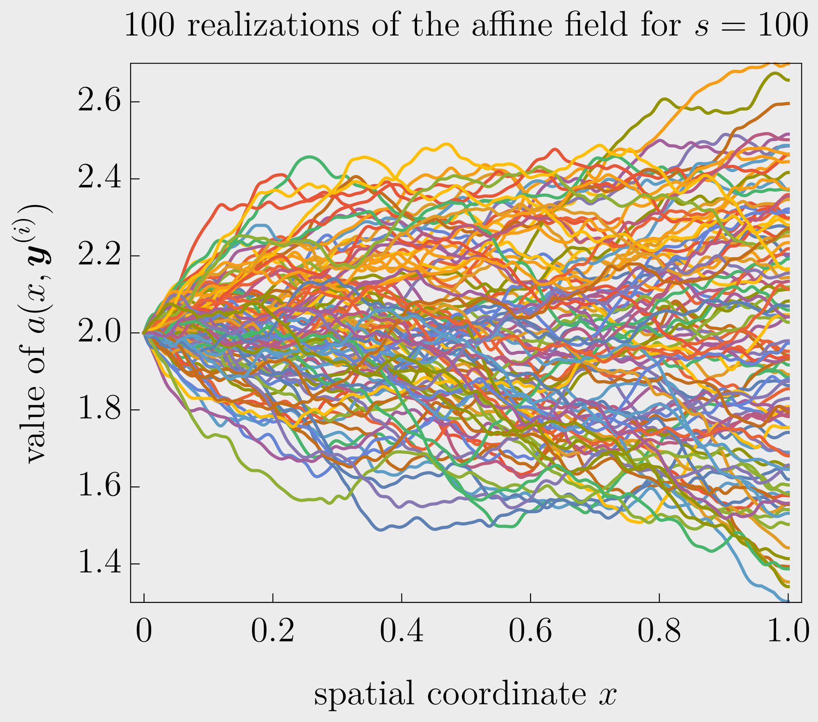
(A6) $\|\boldsymbol{\psi}_i\|_{W^{1,\infty}(D_{\rm ref};\mathbb{R}^d)} \searrow 0$ as $i \to \infty$.

(A7) The reference domain $D_{\rm ref} \subset \mathbb{R}^d$ is a convex, bounded polyhedron.

Affine and uniform model vs. periodic model

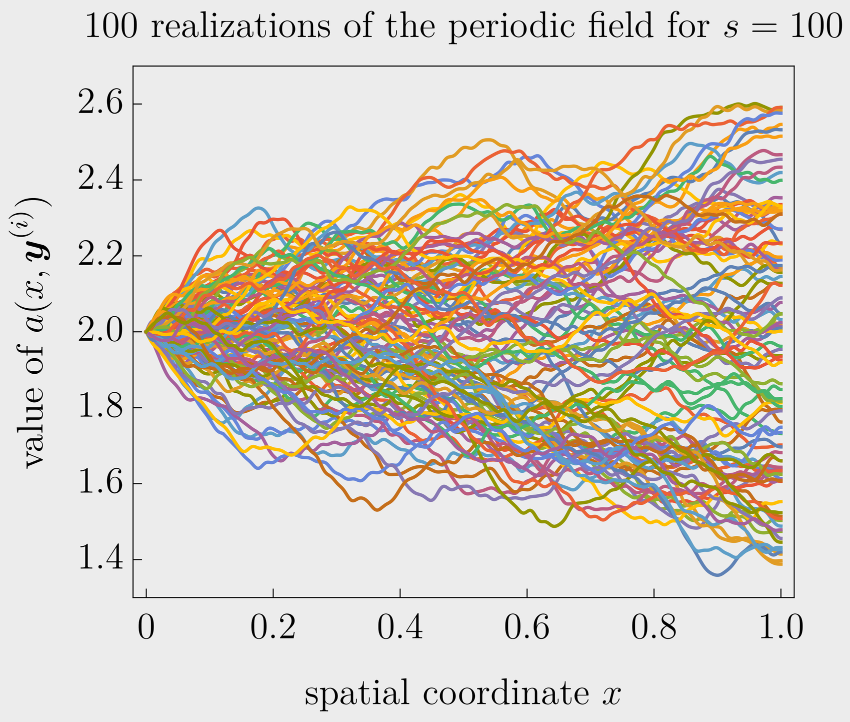
Affine and uniform model

$$a(x, \boldsymbol{y}) = \overline{a}(x) + \sum_{i=1}^{s} y_i \psi_i(x)$$



Periodic model [2]

$$a(x, \boldsymbol{y}) = \overline{a}(x) + \frac{1}{\sqrt{6}} \sum_{i=1}^{s} \sin(2\pi y_i) \psi_i(x)$$



$$\overline{a}(x) = 2, \quad \psi_i(x) = i^{-3/2} \sin((i - \tfrac{1}{2})\pi x), \quad x \in [0,1]$$

The means and covariances of the affine and periodic model coincide.

Main result

For all $\boldsymbol{y} \in U$, we find the transported solution $\widehat{u}(\cdot, \boldsymbol{y}) \in H_0^1(D_{\rm ref})$ in the reference domain such that

$$\widehat{u}(\cdot, \boldsymbol{y}) = u(\boldsymbol{V}(\cdot, \boldsymbol{y}), \boldsymbol{y}) \quad \Leftrightarrow \quad u(\cdot, \boldsymbol{y}) = \widehat{u}(\boldsymbol{V}^{-1}(\cdot, \boldsymbol{y}), \boldsymbol{y}).$$

Let $\widehat{u}_{s,h}(\cdot, \boldsymbol{y}) := \widehat{u}_h(\cdot, (y_1, \ldots, y_s, 0, 0, \ldots))$ denote the *dimensionally-truncated, conforming first order finite element approximation* of $\widehat{u}(\cdot, \boldsymbol{y})$ subject to a regular uniform triangulation of $D_{\rm ref}$. A *rank-1 lattice quasi-Monte Carlo (QMC) rule* is an equal weight cubature rule over the point set

$$\boldsymbol{y}^{(i)} = \mathrm{mod}(\tfrac{i\boldsymbol{z}}{n}, 1) - \tfrac{\boldsymbol{1}}{\boldsymbol{2}}, \quad i \in \{1, \ldots, n\},$$

completely determined by a *generating vector* $\boldsymbol{z} \in \mathbb{N}^s$ and the number of cubature nodes n.

Theorem [1]. *Let $f \in \mathcal{C}^\infty(\mathcal{D})$ be an analytic function. A rank-1 lattice QMC rule can be constructed by a fast component-by-component (CBC) algorithm such that*

$$\left\| \int_U \widehat{u}(\cdot, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} - \frac{1}{n} \sum_{i=1}^{n} \widehat{u}_{s,h}(\cdot, \boldsymbol{y}^{(i)}) \right\|_{L^1(D_{\rm ref})} = \mathcal{O}(s^{-2/p+1} + n^{-1/p} + h^2),$$

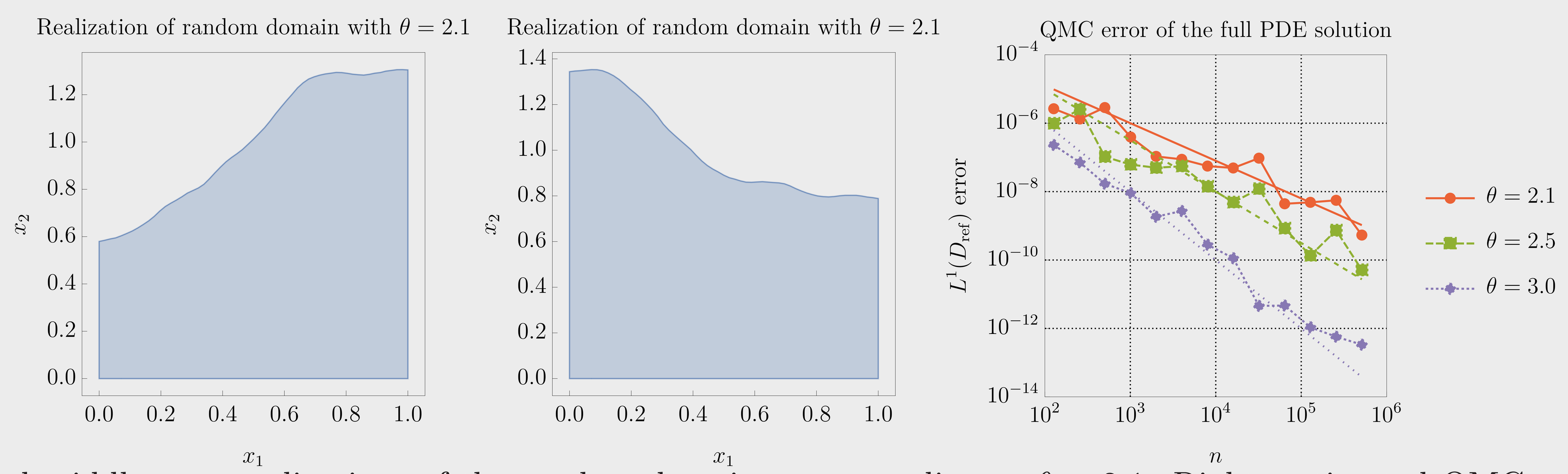
where the implied coefficient is independent of s, n, and the finite element mesh size h.

Numerical experiments

Let the reference domain be the unit square $D_{\rm ref} = (0,1)^2$. We consider the domain parameterization

$$D(\boldsymbol{y}) := \Big\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, \ 0 \leq x_2 \leq 1 + \tfrac{1}{\sqrt{6}} \textstyle\sum_{i=1}^{s} \sin(2\pi y_i) \psi_i(x)\Big\}, \quad \boldsymbol{y} \in [-\tfrac{1}{2}, \tfrac{1}{2}]^s,$$

where only the top edge is uncertain, $\|\psi_i\|_{W^{1,\infty}(D_{\rm ref})} \propto i^{-\theta+1}$, $s = 100$, and $\theta \in \{2.1, 2.5, 3.0\}$.



Left and middle: two realizations of the random domain corresponding to $\theta = 2.1$. Right: estimated QMC cubature errors corresponding to $\theta \in \{2.1, 2.5, 3.0\}$. Increasing θ results in a faster cubature convergence rate.

Conclusions

We analyzed a class of QMC cubature rules used to assess the statistical response of the Poisson problem subject to domain uncertainty. The domain uncertainty was modeled using periodically parameterized random variables. A major advantage of this framework is that it allows us to develop computationally simple QMC rules with higher order cubature convergence rates.

References

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