Lemma 0.1 (Oscillatory integral family)**.**

$$\int_0^1 \cdots \int_0^1 \cos\left(2\pi w_1 + \sum_{i=1}^d c_i x_i\right) \mathrm{d}x_d \cdots \mathrm{d}x_1 = 2^d \cos\left(2\pi w_1 + \frac{1}{2}\sum_{i=1}^d c_i\right) \prod_{i=1}^d \frac{\sin(\frac{c_i}{2})}{c_i}.$$

Proof. The claim can be proved *a fortiori* by first proving the more general integral formula

$$\int_0^1 \cdots \int_0^1 \cos\left(C + \sum_{i=1}^d c_i x_i\right) \mathrm{d}x_d \cdots \mathrm{d}x_1 = 2^d \cos\left(C + \frac{1}{2}\sum_{i=1}^d c_i\right) \prod_{i=1}^d \frac{\sin(\frac{c_i}{2})}{c_i}, \quad C \in \mathbb{R}, \tag{1}$$

using induction with respect to the dimension $d \in \mathbb{Z}_+$. The basis $d = 1$ of the induction argument follows by first computing

$$\begin{aligned}
\int_0^1 \cos(C + c_1 x_1)\,\mathrm{d}x_1 &= \frac{1}{c_1}\big[\sin(C + c_1) - \sin(C)\big] \\
&= \frac{1}{c_1}\big[\cos(C + c_1 - \tfrac{\pi}{2}) - \cos(C - \tfrac{\pi}{2})\big] \\
&= \frac{1}{c_1}\big[\cos(C + \tfrac{1}{2}c_1 + \tfrac{1}{2}c_1 - \tfrac{\pi}{2}) - \cos(C + \tfrac{1}{2}c_1 - \tfrac{1}{2}c_1 - \tfrac{\pi}{2})\big] \\
&= 2\cos(C + \tfrac{1}{2}c_1)\frac{\sin(\frac{c_1}{2})}{c_1},
\end{aligned}$$

where the final step follows from the trigonometric identity $\cos(x+y) - \cos(x-y) = 2\cos(x + \frac{\pi}{2})\sin(y)$ with $x = C + \frac{1}{2}c_1 - \frac{\pi}{2}$ and $y = \frac{1}{2}c_1$.

Suppose that (1) holds for some $d \in \mathbb{Z}_+$. Then it follows that

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 \cos\left(C + \sum_{i=1}^{d+1} c_i x_i\right) \mathrm{d}x_{d+1} \cdots \mathrm{d}x_1 \\
&= \int_0^1 \cdots \int_0^1 \left(\int_0^1 \cos\left(C + \sum_{i=1}^d c_i x_i + c_{d+1}x_{d+1}\right) \mathrm{d}x_{d+1}\right) \mathrm{d}x_d \cdots \mathrm{d}x_1 \\
&= \frac{1}{c_{d+1}} \int_0^1 \cdots \int_0^1 \left[\sin\left(C + \sum_{i=1}^d c_i x_i + c_{d+1}\right) - \sin\left(C + \sum_{i=1}^d c_i x_i\right)\right] \mathrm{d}x_d \cdots \mathrm{d}x_1 \\
&= \frac{1}{c_{d+1}} \int_0^1 \cdots \int_0^1 \left[\cos\left(C + \sum_{i=1}^d c_i x_i + c_{d+1} - \frac{\pi}{2}\right) - \cos\left(C + \sum_{i=1}^d c_i x_i - \frac{\pi}{2}\right)\right] \mathrm{d}x_d \cdots \mathrm{d}x_1 \\
&= \frac{1}{c_{d+1}} \Bigg[2^d \cos\left(C + \frac{1}{2}\sum_{i=1}^d c_i + c_{d+1} - \frac{\pi}{2}\right) \prod_{i=1}^d \frac{\sin(\frac{c_i}{2})}{c_i} \\
&\qquad - 2^d \cos\left(C + \frac{1}{2}\sum_{i=1}^d c_i - \frac{\pi}{2}\right) \prod_{i=1}^d \frac{\sin(\frac{c_i}{2})}{c_i}\Bigg] \\
&= \frac{2^d}{c_{d+1}} \left[\cos\left(C + \frac{1}{2}\sum_{i=1}^d c_i + c_{d+1} - \frac{\pi}{2}\right) - \cos\left(C + \frac{1}{2}\sum_{i=1}^d c_i - \frac{\pi}{2}\right)\right] \prod_{i=1}^d \frac{\sin(\frac{c_i}{2})}{c_i} \\
&= 2^{d+1} \cos\left(C + \frac{1}{2}\sum_{i=1}^{d+1} c_i\right) \prod_{i=1}^{d+1} \frac{\sin(\frac{c_i}{2})}{c_i},
\end{aligned}$$

where the final step follows once again from the trigonometric identity $\cos(x+y) - \cos(x-y) = 2\cos(x + \frac{\pi}{2})\sin(y)$, this time by choosing $x = C + \frac{1}{2}\sum_{i=1}^{d+1} c_i - \frac{\pi}{2}$ and $y = \frac{1}{2}c_{d+1}$. This proves the assertion. □