# Quasi-Monte Carlo methods for optimal control problems subject to parabolic PDE constraints under uncertainty

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## Optimal control problem

We consider the optimal control problem $\min_{z\in\mathcal{Z}} J(z)$, where

$$J(z) := \mathcal{R}\Bigg(\underbrace{\frac{\alpha_1}{2}\int_0^T \|u^{\boldsymbol{y}}(\cdot,t)-\widehat{u}(\cdot,t)\|_V^2\,\mathrm{d}t + \frac{\alpha_2}{2}\|u^{\boldsymbol{y}}(\cdot,T)-\widehat{u}(\cdot,T)\|_{L^2(D)}^2}_{=:\Phi^{\boldsymbol{y}}(z)}\Bigg) + \frac{\alpha_3}{2}\int_0^T \|z(\cdot,t)\|_{V'}^2\,\mathrm{d}t \quad (1)$$

and $\alpha_1,\alpha_2,\alpha_3>0$ subject to the parametric parabolic partial differential equation (PDE) constraint

$$\begin{cases} \partial_t u^{\boldsymbol{y}}(\boldsymbol{x},t) - \nabla\cdot(a^{\boldsymbol{y}}(\boldsymbol{x},t)\nabla u^{\boldsymbol{y}}(\boldsymbol{x},t)) = z(\boldsymbol{x},t), & \boldsymbol{x}\in D,\ t\in I,\\ u^{\boldsymbol{y}}(\boldsymbol{x},t)=0, & \boldsymbol{x}\in\partial D,\ t\in I,\\ u^{\boldsymbol{y}}(\boldsymbol{x},0)=u_0(\boldsymbol{x}), & \boldsymbol{x}\in D \end{cases} \quad (2)$$

for all parameters $\boldsymbol{y}\in[-\frac12,\frac12]^{\mathbb{N}} =: U$, with $V := H_0^1(D)$, $V' = H^{-1}(D)$, $I := [0,T]$, $z\in L^2(V';I)$ is the control, $\widehat{u}\in L^2(V;I)$ is the target state, $u_0\in L^2(D)$ is a known initial heat distribution, and $D\subset\mathbb{R}^d$, $d\in\{1,2,3\}$, is a bounded Lipschitz domain. The set of admissible controls $\varnothing\neq\mathcal{Z}\subseteq L^2(V';I)$ is either $\mathcal{Z}=L^2(V';I)$ or a bounded, closed, and convex set. We consider

(R1) the *risk neutral* expected value risk measure $\mathcal{R}(\cdot) := \int_U \cdot\,\mathrm{d}\boldsymbol{y}$;

(R2) the *risk averse* entropic risk measure $\mathcal{R}(\cdot) := \frac{1}{\theta}\ln\left(\int_U \mathrm{e}^{\theta\,\cdot}\,\mathrm{d}\boldsymbol{y}\right)$ for some $\theta>0$.

## Random coefficient

The random coefficient $a^{\boldsymbol{y}}(\boldsymbol{x},t)$ in (2) is assumed to have affine dependence on the uncertain variables, and it is modeled by a series expansion

$$a^{\boldsymbol{y}}(\boldsymbol{x},t) = a_0(\boldsymbol{x},t) + \sum_{j=1}^{\infty} y_j\psi_j(\boldsymbol{x},t) \quad (3)$$

for $\boldsymbol{x}\in D$, $\boldsymbol{y}\in U$, and $t\in I$, where we assume the following:

(i) for all $t\in I$, it holds that $a_0(\cdot,t)\in L^\infty(D)$ and $\psi_j(\cdot,t)\in L^\infty(D)$ for all $j\geq 1$;

(ii) $(\sup_{t\in I}\|\psi_j(\cdot,t)\|_{L^\infty(D)})_{j=1}^\infty \in \ell^p$ for some $p\in(0,1)$;

(iii) $t\mapsto a^{\boldsymbol{y}}(\boldsymbol{x},t)$ is measurable on $I$;

(iv) there exist constants $a_{\min}$ and $a_{\max}$ such that $0<a_{\min}\leq a^{\boldsymbol{y}}(\boldsymbol{x},t)\leq a_{\max}<\infty$ for all $\boldsymbol{x}\in D$, $t\in I$, and $\boldsymbol{y}\in U$.

## Optimality system

Let $\mathcal{R}$ be the expected value or the entropic risk measure. A control $z^*$ is the unique minimizer of the problem (1)–(2) if and only if it satisfies the optimality system

$$\begin{cases} \partial_t u^{\boldsymbol{y}}(\boldsymbol{x},t) - \nabla\cdot(a^{\boldsymbol{y}}(\boldsymbol{x},t)\nabla u^{\boldsymbol{y}}(\boldsymbol{x},t)) = z^*(\boldsymbol{x},t),\\ u^{\boldsymbol{y}}(\cdot,t)|_{\partial D}=0,\\ u^{\boldsymbol{y}}(\boldsymbol{x},0)=u_0(\boldsymbol{x}),\\ -\partial_t q^{\boldsymbol{y}}(\boldsymbol{x},t) - \nabla\cdot(a^{\boldsymbol{y}}(\boldsymbol{x},t)\nabla q^{\boldsymbol{y}}(\boldsymbol{x},t))\\ \qquad\qquad = \alpha_1(u^{\boldsymbol{y}}(\boldsymbol{x},t)-\widehat{u}(\boldsymbol{x},t)),\\ q^{\boldsymbol{y}}(\cdot,t)|_{\partial D}=0,\\ q^{\boldsymbol{y}}(\boldsymbol{x},T)=\alpha_2(u^{\boldsymbol{y}}(\boldsymbol{x},T)-\widehat{u}(\boldsymbol{x},T)),\\ z^*\in\mathcal{Z},\\ \langle J'(z^*),z-z^*\rangle_{L^2(V;I),L^2(V';I)}\geq 0 \quad \forall z\in\mathcal{Z}, \end{cases}$$

for $\boldsymbol{x}\in D$, $t\in I$, and $\boldsymbol{y}\in U$.

In the case (R1), the Fréchet derivative of $J$ is

$$J'(z) = \int_U q^{\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y} + \alpha_3(-\Delta)^{-1}z$$

and in the case (R2), the Fréchet derivative is

$$J'(z) = \frac{\int_U \mathrm{e}^{\theta\Phi^{\boldsymbol{y}}(z)}q^{\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}}{\int_U \mathrm{e}^{\theta\Phi^{\boldsymbol{y}}(z)}\,\mathrm{d}\boldsymbol{y}} + \alpha_3(-\Delta)^{-1}z.$$

## Main result

The series (3) is truncated to $s$ terms and the $s$-dimensional integrals appearing in $J$ and $J'$ are replaced with randomly shifted lattice quasi-Monte Carlo (QMC) rules with $n$ cubature nodes

$$\boldsymbol{y}^{(i)} = \mathrm{mod}\left(\tfrac{i\boldsymbol{z}}{n}+\boldsymbol{\Delta},1\right)-\tfrac{\boldsymbol{1}}{\boldsymbol{2}},\quad i\in\{1,\ldots,n\},$$

where $\boldsymbol{z}\in\mathbb{N}^s$ is a *generating vector* and $\boldsymbol{\Delta}\in[0,1]^s$ is a uniform random shift. The discretized objective remains *convex*. Let $z_{s,n}^*$ be the optimal control of the discretized optimization problem.

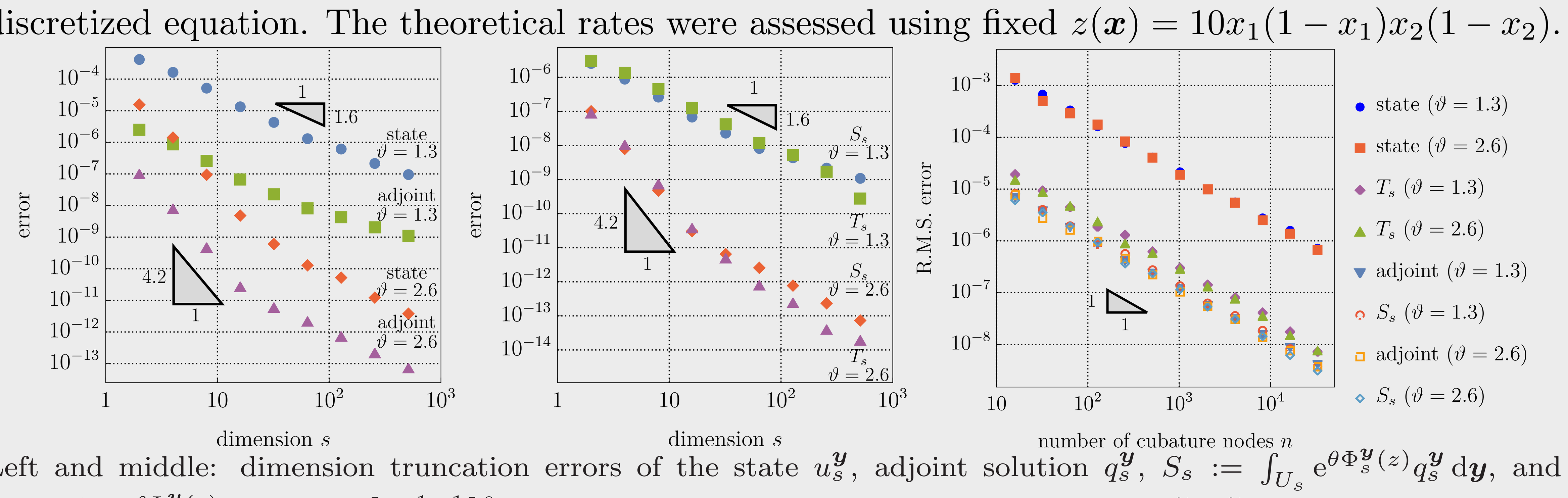
**Theorem** [1]. *Let $\mathcal{R}$ be the expected value or the entropic risk measure. A randomly shifted rank-1 lattice QMC rule can be constructed by a fast component-by-component (CBC) algorithm such that*

$$\sqrt{\mathbb{E}_{\boldsymbol{\Delta}}\|z^*-z_{s,n}^*\|_{L^2(V';I)}^2} = \mathcal{O}(s^{-2/p+1}+n^{-\min\{1/p-1/2,1-\delta\}})\quad \forall\delta>0,$$

*where the implied coefficient is independent of $s$ and $n$.*

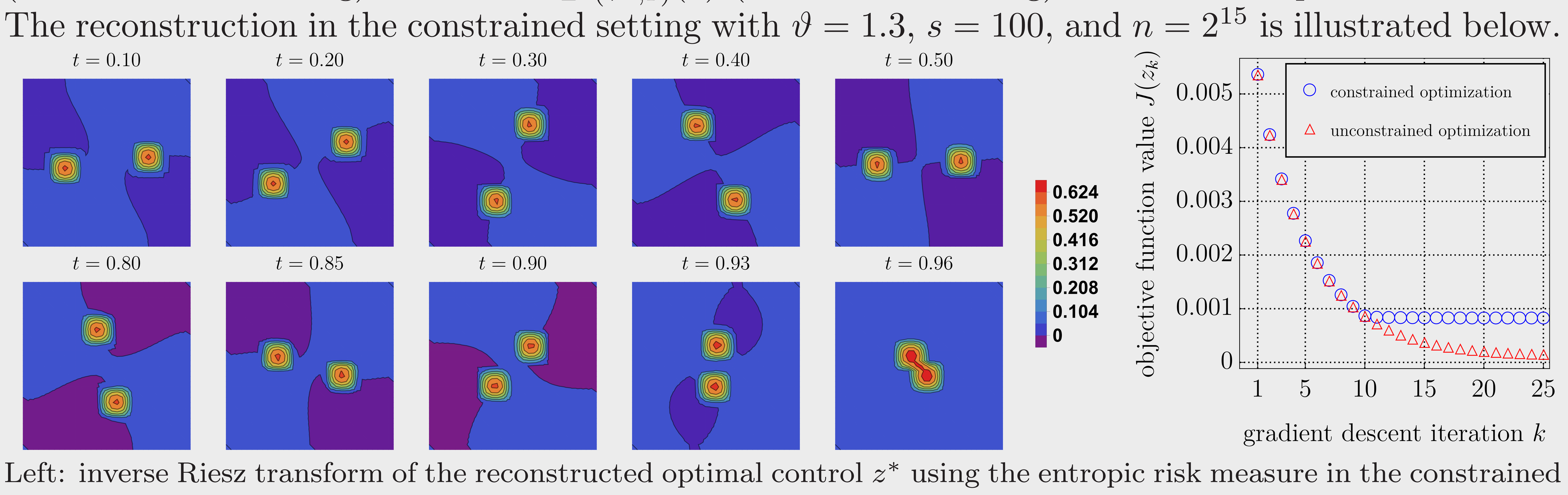
## Numerical experiments

Let $D=(0,1)^2$, $T=1$, $\theta=10$, and let $a^{\boldsymbol{y}}$ be time-independent with $\psi_j$ such that $\|\psi_j\|_{L^\infty(D)}\propto j^{-\vartheta}$. A first order finite element method with mesh size $h=2^{-5}$ was used to discretize the PDE spatially and the implicit Euler method with step size $\Delta t = 2\cdot 10^{-3}$ was used to solve the resulting semi-discretized equation. The theoretical rates were assessed using fixed $z(\boldsymbol{x}) = 10x_1(1-x_1)x_2(1-x_2)$.




Left and middle: dimension truncation errors of the state $u_s^{\boldsymbol{y}}$, adjoint solution $q_s^{\boldsymbol{y}}$, $S_s := \int_{U_s}\mathrm{e}^{\theta\Phi_s^{\boldsymbol{y}}(z)}q_s^{\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}$, and $T_s := \int_{U_s}\mathrm{e}^{\theta\Phi_s^{\boldsymbol{y}}(z)}\,\mathrm{d}\boldsymbol{y}$, $U_s := [-\frac12,\frac12]^s$. Right: QMC errors of dimensionally-truncated $u_s^{\boldsymbol{y}}$, $q_s^{\boldsymbol{y}}$, $S_s$, and $T_s$, $s=100$.

We considered recovering the optimal control $z^*$ given a certain target state $\widehat{u}$ with $\mathcal{Z}=L^2(V';I)$ (unconstrained setting) and $\mathcal{Z}=\overline{B_{L^2(V';I)}}(2)$ (constrained setting) with the entropic risk measure. The reconstruction in the constrained setting with $\vartheta=1.3$, $s=100$, and $n=2^{15}$ is illustrated below.




Left: inverse Riesz transform of the reconstructed optimal control $z^*$ using the entropic risk measure in the constrained setting. Right: objective function values for each gradient descent iteration in constrained and unconstrained settings.

## Conclusions

QMC discretization preserves the convexity of (1) unlike, e.g., sparse grids which have negative weights, while exhibiting faster-than-Monte Carlo convergence rates independently of the dimension $s$.

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## References

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