Introduction to X-ray tomography

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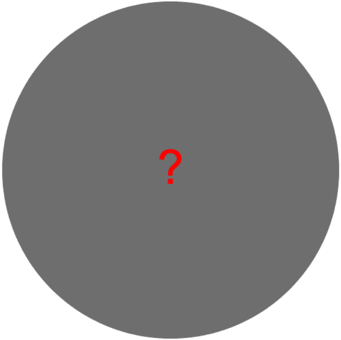
The content of this lecture follows roughly the material presented in the following monographs.

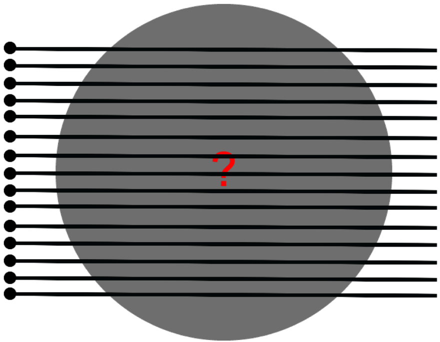
- J. Kaipio and E. Somersalo. Statistical and Computational Inverse Problems. 2005.
- J. L. Mueller and S. Siltanen. Linear and Nonlinear Inverse Problems with Practical Applications. 2012.

ASTRA Toolbox for 2D and 3D tomography:
`https://www.astra-toolbox.com/`

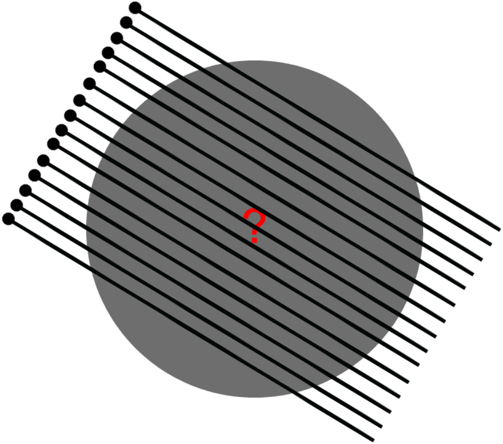
Parallel-beam X-ray tomography



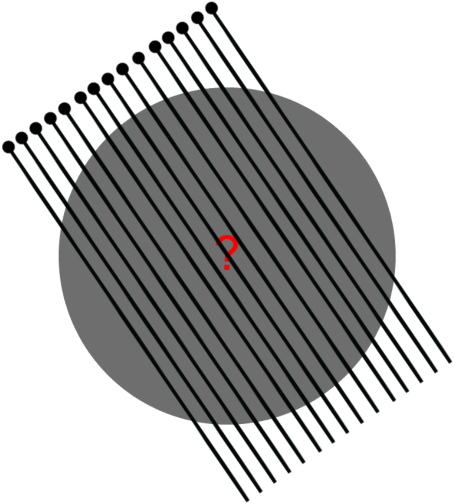
Parallel-beam X-ray tomography



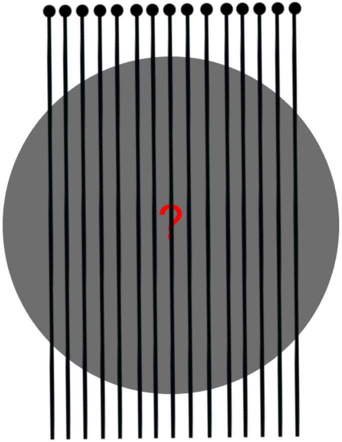
Parallel-beam X-ray tomography



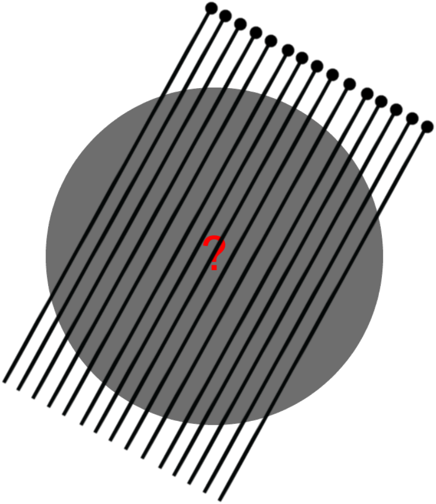
Parallel-beam X-ray tomography



Parallel-beam X-ray tomography



Parallel-beam X-ray tomography



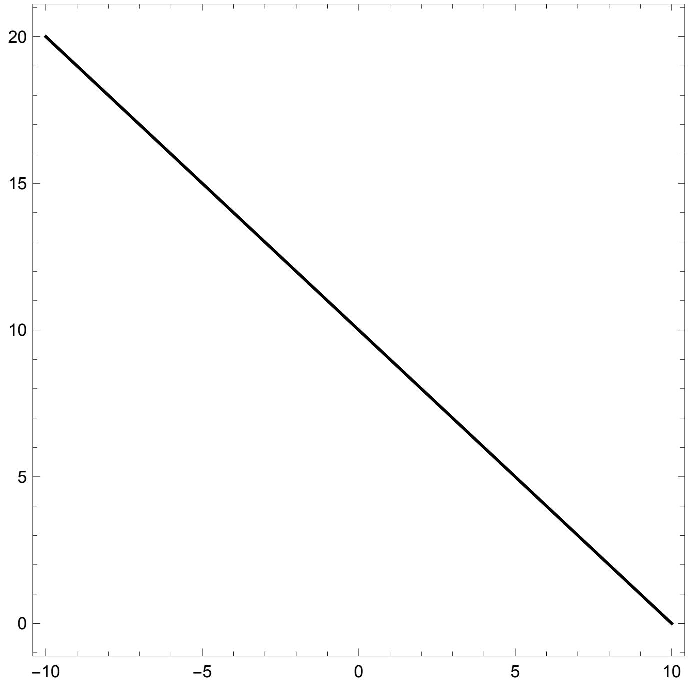
Radon transform in $\mathbb{R}^2$

Let L be a straight line in $\mathbb{R}^2$.

Any line in $\mathbb{R}^2$ can be parameterized as

$$L = \{s\omega + t\omega^\perp;\ t \in \mathbb{R}\} \quad \text{for some } s \in \mathbb{R} \text{ and } \omega \in S^1,$$

where $\omega^\perp \perp \omega$.


20
15
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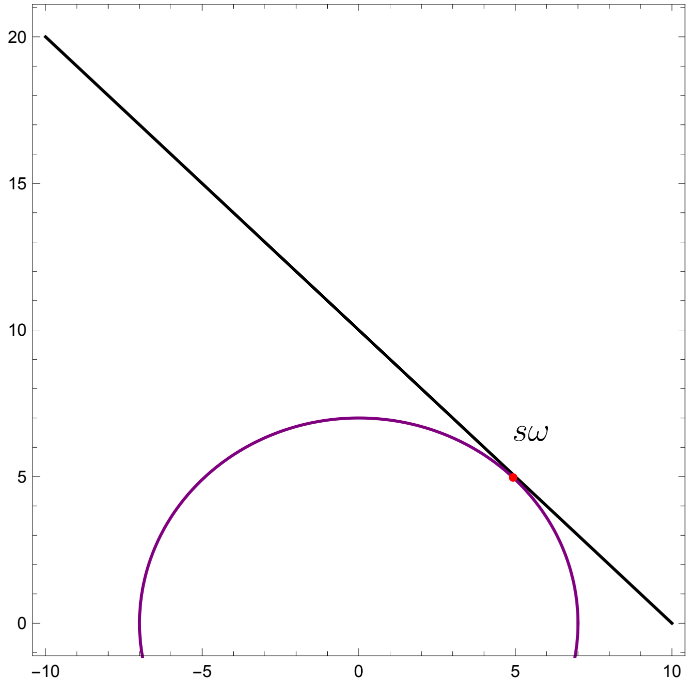
10



20
15
10
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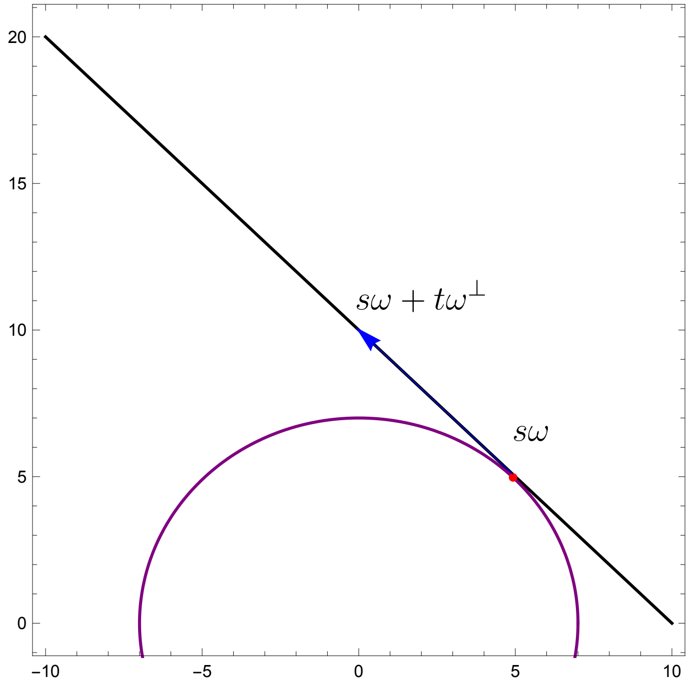
0
−10
−5
0
5
10
$s\omega$



20
15
10
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0
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−5
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5
10
$s\omega + t\omega^{\perp}$
$s\omega$

Radon transform in $\mathbb{R}^2$

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where $\omega^\perp \perp \omega$.

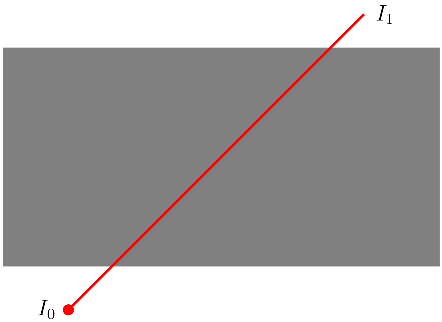
Writing $\omega = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, we get

$$L = L(s,\theta) = \left\{ s \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + t \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix};\ t \in \mathbb{R} \right\}, \quad s \in \mathbb{R} \text{ and } \theta \in [0,\pi).$$

The *Radon transform* of a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ on L is defined as

$$\mathcal{R}f(L) = \int_L f(\boldsymbol{x})\,|\mathrm{d}\boldsymbol{x}| = \int_{-\infty}^{\infty} f(s\cos\theta + t\sin\theta, s\sin\theta - t\cos\theta)\,\mathrm{d}t.$$

Let f be a nonnegative function modeling X-ray attenuation (density) inside a physical body.



Beer–Lambert law:

$$\mathcal{R}f(L) = \log \frac{I_0}{I_1}.$$

$f_{10,1}$	$f_{10,2}$	$f_{10,3}$	$f_{10,4}$	$f_{10,5}$	$f_{10,6}$	$f_{10,7}$	$f_{10,8}$	$f_{10,9}$	$f_{10,10}$
$f_{9,1}$	$f_{9,2}$	$f_{9,3}$	$f_{9,4}$	$f_{9,5}$	$f_{9,6}$	$f_{9,7}$	$f_{9,8}$	$f_{9,9}$	$f_{9,10}$
$f_{8,1}$	$f_{8,2}$	$f_{8,3}$	$f_{8,4}$	$f_{8,5}$	$f_{8,6}$	$f_{8,7}$	$f_{8,8}$	$f_{8,9}$	$f_{8,10}$
$f_{7,1}$	$f_{7,2}$	$f_{7,3}$	$f_{7,4}$	$f_{7,5}$	$f_{7,6}$	$f_{7,7}$	$f_{7,8}$	$f_{7,9}$	$f_{7,10}$
$f_{6,1}$	$f_{6,2}$	$f_{6,3}$	$f_{6,4}$	$f_{6,5}$	$f_{6,6}$	$f_{6,7}$	$f_{6,8}$	$f_{6,9}$	$f_{6,10}$
$f_{5,1}$	$f_{5,2}$	$f_{5,3}$	$f_{5,4}$	$f_{5,5}$	$f_{5,6}$	$f_{5,7}$	$f_{5,8}$	$f_{5,9}$	$f_{5,10}$
$f_{4,1}$	$f_{4,2}$	$f_{4,3}$	$f_{4,4}$	$f_{4,5}$	$f_{4,6}$	$f_{4,7}$	$f_{4,8}$	$f_{4,9}$	$f_{4,10}$
$f_{3,1}$	$f_{3,2}$	$f_{3,3}$	$f_{3,4}$	$f_{3,5}$	$f_{3,6}$	$f_{3,7}$	$f_{3,8}$	$f_{3,9}$	$f_{3,10}$
$f_{2,1}$	$f_{2,2}$	$f_{2,3}$	$f_{2,4}$	$f_{2,5}$	$f_{2,6}$	$f_{2,7}$	$f_{2,8}$	$f_{2,9}$	$f_{2,10}$
$f_{1,1}$	$f_{1,2}$	$f_{1,3}$	$f_{1,4}$	$f_{1,5}$	$f_{1,6}$	$f_{1,7}$	$f_{1,8}$	$f_{1,9}$	$f_{1,10}$

Let us consider the computational domain $[-1,1]^2$. We divide this region into $n \times n$ pixels and approximate the density by a piecewise constant function with constant value

$$f_{i,j} \text{ in pixel } P_{i,j}$$

for $i, j \in \{1, \dots, n\}$.

$$P_{i,j} := \{(x,y);\ -1 + 2\,\tfrac{j-1}{n} < x < -1 + 2\,\tfrac{j}{n},\ -1 + 2\,\tfrac{i-1}{n} < y < -1 + 2\,\tfrac{i}{n}\}$$

x_{91}	x_{92}	x_{93}	x_{94}	x_{95}	x_{96}	x_{97}	x_{98}	x_{99}	x_{100}
x_{81}	x_{82}	x_{83}	x_{84}	x_{85}	x_{86}	x_{87}	x_{88}	x_{89}	x_{90}
x_{71}	x_{72}	x_{73}	x_{74}	x_{75}	x_{76}	x_{77}	x_{78}	x_{79}	x_{80}
x_{61}	x_{62}	x_{63}	x_{64}	x_{65}	x_{66}	x_{67}	x_{68}	x_{69}	x_{70}
x_{51}	x_{52}	x_{53}	x_{54}	x_{55}	x_{56}	x_{57}	x_{58}	x_{59}	x_{60}
x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	x_{46}	x_{47}	x_{48}	x_{49}	x_{50}
x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	x_{36}	x_{37}	x_{38}	x_{39}	x_{40}
x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}	x_{28}	x_{29}	x_{30}
x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}

It is convenient to reshape the matrix/image $(f_{i,j})$ into a vector x of length n^2 so that

$$x_{(j-1)n+i} = f_{i,j}, \quad i, j \in \{1, \ldots, n\}.$$

The image on the left illustrates the new numbering corresponding to the pixels.

Note that `x = f(:)` and `f = reshape(x,n,n)`.

Measurement model

Let us consider a measurement setup where we take X-ray measurements of an object using X-rays $L(s_1, \theta), \dots, L(s_K, \theta)$ taken at angles $\theta \in \{\theta_1, \dots, \theta_M\}$. The total number of X-rays is $Q = MK$.

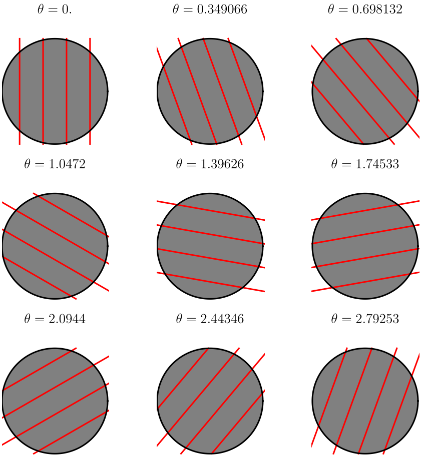
For brevity, let us write $L_{(m-1)K+k} := L(s_k, \theta_m)$ for $k \in \{1, \dots, K\}$ and $m \in \{1, \dots, M\}$.

The measurement model is

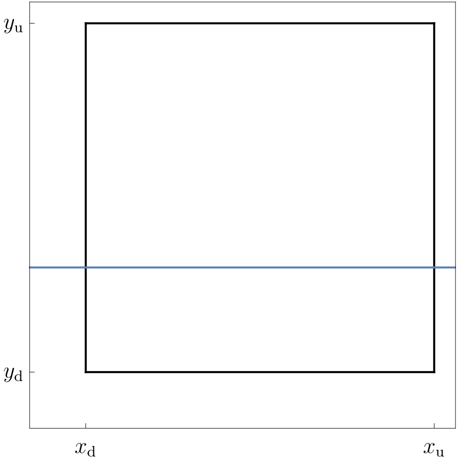
$$y = \begin{bmatrix} \int_{L_1} f(\boldsymbol{x}) |\mathrm{d}\boldsymbol{x}| \\ \vdots \\ \int_{L_Q} f(\boldsymbol{x}) |\mathrm{d}\boldsymbol{x}| \end{bmatrix} + \varepsilon \approx \begin{bmatrix} \sum_{j=1}^{n^2} A_{1,j} x_j \\ \vdots \\ \sum_{j=1}^{n^2} A_{Q,j} x_j \end{bmatrix} + \varepsilon = Ax + \varepsilon,$$

where $A \in \mathbb{R}^{Q \times n^2}$ and $A_{i,j}$ is the distance that ray L_i travels through pixel j. Here, x is a vector containing the (piecewise constant) densities within each pixel and ε is measurement noise.

$$L_{(m-1)K+k} = \left\{ s_k \begin{bmatrix} \cos\theta_m \\ \sin\theta_m \end{bmatrix} + t \begin{bmatrix} \sin\theta_m \\ -\cos\theta_m \end{bmatrix};\ t \in \mathbb{R} \right\}, \quad \begin{matrix} k = 1, \ldots, K, \\ m = 1, \ldots, M. \end{matrix}$$



Pixel-by-pixel construction of the tomography matrix A



Case $\cos\theta = 0$ and $\sin\theta = 1$:

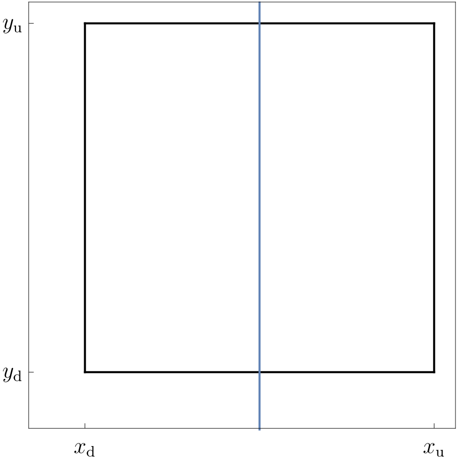
$$\begin{bmatrix} x_d \\ y_d \end{bmatrix} \leq \begin{bmatrix} s\cos\theta + t\sin\theta \\ s\sin\theta - t\cos\theta \end{bmatrix} < \begin{bmatrix} x_u \\ y_u \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} x_d \\ y_d \end{bmatrix} \leq \begin{bmatrix} t \\ s \end{bmatrix} < \begin{bmatrix} x_u \\ y_u \end{bmatrix}.$$

The distance that ray L_m travels through pixel k is

$$A_{m,k} = \int_{L_m} \chi_k \, |\mathrm{d}\boldsymbol{x}| = \int\limits_{\substack{x_d < t \leq x_u \\ y_d \leq s < y_u}} \mathrm{d}t = \begin{cases} x_u - x_d & \text{if } y_d \leq s < y_u, \\ 0 & \text{otherwise.} \end{cases}$$

N.B. In here and in the following, $\chi_k = \chi_k(\boldsymbol{x})$ denotes the characteristic function of the k^{th} pixel. In the above illustration, the pixel is denoted by the rectangle $[x_d, x_u) \times [y_d, y_u)$.



Case $\cos\theta = 1$ and $\sin\theta = 0$:

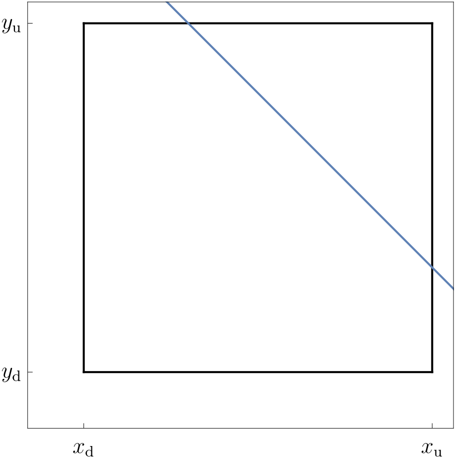
$$\begin{bmatrix} x_d \\ y_d \end{bmatrix} \le \begin{bmatrix} s\cos\theta + t\sin\theta \\ s\sin\theta - t\cos\theta \end{bmatrix} < \begin{bmatrix} x_u \\ y_u \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} x_d \\ y_d \end{bmatrix} \le \begin{bmatrix} s \\ -t \end{bmatrix} < \begin{bmatrix} x_u \\ y_u \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} x_d \\ -y_u \end{bmatrix} < \begin{bmatrix} s \\ t \end{bmatrix} \le \begin{bmatrix} x_u \\ -y_d \end{bmatrix}.$$

The distance that ray L_m travels through pixel k is

$$A_{m,k} = \int_{L_m} \chi_k \, |\mathrm{d}\boldsymbol{x}| = \int\limits_{\substack{-y_u < t \le -y_d \\ x_d < s \le x_u}} \mathrm{d}t = \begin{cases} y_u - y_d & \text{if } x_d < s \le x_u, \\ 0 & \text{otherwise.} \end{cases}$$



Case $\cos\theta > 0$:

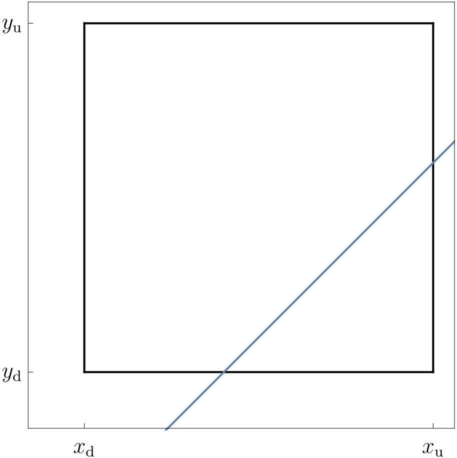
$$\begin{bmatrix} x_\mathrm{d} \\ y_\mathrm{d} \end{bmatrix} < \begin{bmatrix} s\cos\theta + t\sin\theta \\ s\sin\theta - t\cos\theta \end{bmatrix} < \begin{bmatrix} x_\mathrm{u} \\ y_\mathrm{u} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \frac{x_\mathrm{d} - s\cos\theta}{\sin\theta} \\ \frac{s\sin\theta - y_\mathrm{u}}{\cos\theta} \end{bmatrix} < \begin{bmatrix} t \\ t \end{bmatrix} < \begin{bmatrix} \frac{x_\mathrm{u} - s\cos\theta}{\sin\theta} \\ \frac{s\sin\theta - y_\mathrm{d}}{\cos\theta} \end{bmatrix}.$$

The distance that ray L_m travels through pixel k is

$$A_{m,k} = \int_{L_m} \chi_k \, |\mathrm{d}\boldsymbol{x}| = \int_{\max\left\{\frac{x_\mathrm{d} - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_\mathrm{u}}{\cos\theta}\right\} < t < \min\left\{\frac{x_\mathrm{u} - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_\mathrm{d}}{\cos\theta}\right\}} \mathrm{d}t$$

$$= \left(\min\left\{ \frac{x_\mathrm{u} - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_\mathrm{d}}{\cos\theta} \right\} - \max\left\{ \frac{x_\mathrm{d} - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_\mathrm{u}}{\cos\theta} \right\} \right)_+ .$$



Case $\cos\theta < 0$:

$$\begin{bmatrix} x_{\mathrm{d}} \\ y_{\mathrm{d}} \end{bmatrix} < \begin{bmatrix} s\cos\theta + t\sin\theta \\ s\sin\theta - t\cos\theta \end{bmatrix} < \begin{bmatrix} x_{\mathrm{u}} \\ y_{\mathrm{u}} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \frac{x_{\mathrm{d}} - s\cos\theta}{\sin\theta} \\ s\sin\theta - y_{\mathrm{u}} \end{bmatrix} < \begin{bmatrix} t \\ t\cos\theta \end{bmatrix} < \begin{bmatrix} \frac{x_{\mathrm{u}} - s\cos\theta}{\sin\theta} \\ s\sin\theta - y_{\mathrm{d}} \end{bmatrix}$$

$$\overset{!}{\Leftrightarrow} \begin{bmatrix} \frac{x_{\mathrm{d}} - s\cos\theta}{\sin\theta} \\ \frac{s\sin\theta - y_{\mathrm{d}}}{\cos\theta} \end{bmatrix} < \begin{bmatrix} t \\ t \end{bmatrix} < \begin{bmatrix} \frac{x_{\mathrm{u}} - s\cos\theta}{\sin\theta} \\ \frac{s\sin\theta - y_{\mathrm{u}}}{\cos\theta} \end{bmatrix}.$$

The distance that ray L_m travels through pixel k is

$$A_{m,k} = \int_{L_m} \chi_k \, |\mathrm{d}\boldsymbol{x}| = \int_{\max\left\{\frac{x_{\mathrm{d}} - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_{\mathrm{d}}}{\cos\theta}\right\} < t < \min\left\{\frac{x_{\mathrm{u}} - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_{\mathrm{u}}}{\cos\theta}\right\}} \mathrm{d}t$$

$$= \left(\min\left\{ \frac{x_{\mathrm{u}} - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_{\mathrm{u}}}{\cos\theta} \right\} - \max\left\{ \frac{x_{\mathrm{d}} - s\cos\theta}{\sin\theta}, \frac{s\sin\theta - y_{\mathrm{d}}}{\cos\theta} \right\} \right)_+ .$$

Discussion

Tomography problems can be classified into three brackets based on the nature of the measurement data:

- Full angle tomography
  - Sufficient number of measurements from all angles $\rightarrow$ not a very ill-posed problem.
- Limited angle tomography
  - Data collected from a restricted angle of view $\rightarrow$ reconstructions very sensitive to measurement error and it is not possible to reconstruct the object perfectly (even with noiseless data). Applications include, e.g., dental imaging.
- Sparse data tomography
  - The data consist of only a few projection images, possibly from any direction $\rightarrow$ extremely ill-posed inverse problem and prior knowledge necessary for successful reconstructions. (E.g., minimizing a patient's radiation dose.)