Uncertainty Quantification and Quasi-Monte Carlo

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Recap

We considered real Hilbert spaces, which are inner product spaces $(H, \langle \cdot, \cdot \rangle)$ that are complete w.r.t. the induced norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$.

Proposition (Orthogonal decomposition)

If M is a closed subspace of a real Hilbert space H, then

$$H = M \oplus M^{\perp},$$

which means that every element $y \in H$ can be uniquely represented as

$$y = x + x^{\perp}, \quad x \in M, \; x^{\perp} \in M^{\perp}.$$

This decompositions will be very useful for our purposes. For example, for any closed subspace, we can introduce a mapping $P_M \colon H \to M$, $y \mapsto x$, called an *orthogonal projection*.

Lemma

Let $M \subset H$ be a closed subspace. The mapping $P_M : H \to M$, $y \mapsto x$, is an orthogonal projection, i.e., $P_M^2 = P_M$ and $\mathrm{Ran}(P_M) \perp \mathrm{Ran}(I - P_M)$. It satisfies the following properties:

- *P_M is linear;*
- *$\|P_M\| = 1$ if $M \neq \{0\}$;*
- *$I - P_M = P_{M^\perp}$;*
- *$\|y - P_M y\| \leq \|y - z\|$ for all $z \in M$;*
- *$y \in M \;\Rightarrow\; P_M y = y,\; (I - P_M)y = 0$;*
  $y \in M^\perp \Rightarrow\; P_M y = 0,\; (I - P_M)y = y$;
- *$\|y\|^2 = \|P_M y\|^2 + \|(I - P_M)y\|^2$ (Pythagoras).*

Proof. Omitted; see for example [Rudin, Real and Complex Analysis, pp. 34–35]. □

Example

Let H_1 and H_2 be real Hilbert spaces and let $A\colon H_1 \to H_2$ be a continuous linear operator.

The kernel (or null space) of operator A is defined as

$$\mathrm{Ker}(A) := \{x \in H_1 \mid Ax = 0\}.$$

The range (or image) of operator A is defined as

$$\mathrm{Ran}(A) := \{y \in H_2 \mid y = Ax,\ x \in H_1\}.$$

Then we have the following:

- Ker(A) is a *closed* subspace of H_1, and Ran(A) is a subspace of H_2.
- $H_1 = \mathrm{Ker}(A) \oplus (\mathrm{Ker}(A))^{\perp}$.
- $H_2 = \overline{\mathrm{Ran}(A)} \oplus (\mathrm{Ran}(A))^{\perp}$.

We denote

$$\mathcal{L}(X, Y) := \{A \mid A\colon X \to Y \text{ is bounded and linear}\}.$$

Proposition

If Y is complete, then $\mathcal{L}(X, Y)$ is complete w.r.t. operator norm (i.e., it is a Banach space).

Proof. Let $x \in X$ and assume that $A_k \in \mathcal{L}(X, Y)$, $k \in \mathbb{N}$, is a Cauchy sequence. Then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$m, n > N \quad \Rightarrow \quad \|A_m - A_n\| < \frac{\varepsilon}{\|x\|_X}.$$

Especially,

$$\|A_m x - A_n x\|_Y \leq \|A_m - A_n\| \|x\|_X < \varepsilon \quad \text{when } m, n > N,$$

so $(A_k x)$ is a Cauchy sequence in Y and therefore the limit

$$A(x) := \lim_{k\to\infty} A_k x$$

exists.

It is easy to see that $A(x) := \lim_{k\to\infty} A_k x$ is linear. It is also bounded: there exists $N \in \mathbb{N}$ such that

$$m, n > N \quad \Rightarrow \quad \|A_m - A_n\| < 1.$$

Fix $m > N$. Then for all $n > m$,

$$\|A_n\| < 1 + \|A_m\|$$

and thus

$$\|A_n x\|_Y \leq (1 + \|A_m\|)\|x\|_X.$$

But $\|Ax\|_Y = \lim_{n\to\infty} \|A_n x\|_Y \leq (1 + \|A_m\|)\|x\|_X$. Therefore A is bounded.
Finally, we need to show that $\|A_n - A\| \to 0$ as $n \to \infty$. Since we assumed $(A_k)_{k=1}^{\infty}$ to be Cauchy, let $\varepsilon > 0$ be s.t. for $m, n > N$, there holds $\|A_m - A_n\| < \varepsilon$. Then

$$\|(A - A_n)x\|_Y = \lim_{m\to\infty} \|A_m x - A_n x\|_Y \leq \varepsilon\|x\|_X \quad \text{for all } x \in X$$
$$\Rightarrow \quad \|A - A_n\| < \varepsilon.$$

Hence $\|A - A_n\| \to 0$ as $n \to \infty$. □

If $X = H_1$ and $Y = H_2$ are Hilbert spaces, then $\mathcal{L}(H_1, H_2)$ is a complete normed space.

Definition

Let H be a Hilbert space. The space $H' := \mathcal{L}(H, \mathbb{R})$ is called the *topological dual space* of H.

Corollary

If H is a Hilbert space, then H' is complete w.r.t. the operator norm.

Proof. This is an immediate consequence of the previous proposition since $\mathbb{R}$ is a complete Hilbert space. □

Remark. In general, $\mathcal{L}(H_1, H_2)$ is *not* a Hilbert space even when both H_1 and H_2 are. However, in the special case $H' = \mathcal{L}(H, \mathbb{R})$ it turns out that indeed one can associate an inner product that induces the operator norm $\| \cdot \|$ – meaning that H' is a Hilbert space! This is made possible by the *Riesz representation theorem*.

Existence results

Proposition (Riesz representation theorem)

Let H be a real Hilbert space. If $A\colon H \to \mathbb{R}$ is a bounded linear functional, i.e., A is linear and there exists $C > 0$ such that

$$|A(x)| \leq C\|x\| \quad \text{for all } x \in H,$$

then there exists a unique *$y \in H$ such that*

$$A(x) = \langle x, y \rangle \quad \text{for all } x \in H.$$

Proof. If $A \equiv 0$, then $y = 0$ and this is unique. Suppose $A \neq 0$ and let

$$M := \mathrm{Ker}(A) = \{x \in H \mid A(x) = 0\}.$$

Since A is continuous, M is a *closed* subspace of H. Furthermore, by the orthogonal decomposition $H = M \oplus M^{\perp}$, our assumption $A \neq 0$ implies that $M \neq H \Rightarrow M^{\perp} \neq \{0\}$.

Let $x \in H$ and $z \in M^{\perp}$ with $\|z\| = 1$. Define

$$u := A(x)z - A(z)x.$$

Then

$$A(u) = A(x)A(z) - A(z)A(x) = 0.$$

meaning that $u \in M$. In particular $\langle u, z \rangle = \langle A(x)z - A(z)x, z \rangle = 0$ and

$$\begin{aligned} A(x) = A(x) \underbrace{\langle z, z \rangle}_{=\|z\|^2=1} &= \langle A(x)z, z \rangle \\ &= \langle A(z)x, z \rangle = A(z)\langle x, z \rangle = \langle x, zA(z) \rangle. \end{aligned}$$

$\therefore$ The element $y = zA(z)$ satisfies $A(x) = \langle x, y \rangle$.
To prove uniqueness, suppose that there exist $y_1, y_2 \in H$ such that

$$A(x) = \langle x, y_1 \rangle = \langle x, y_2 \rangle.$$

Then $\langle x, y_1 - y_2 \rangle = 0$ for all $x \in H$. Choose $x = y_1 - y_2$. Then

$$0 = \langle y_1 - y_2, y_1 - y_2 \rangle = \|y_1 - y_2\|^2 \quad \Leftrightarrow \quad y_1 = y_2.$$

□

The Riesz operator

Let $x \in H$ and consider the linear mapping $f_x\colon H \to \mathbb{R}$, $z \mapsto \langle z, x\rangle_H$. Note that $f_x \in H'$ since it follows from the Cauchy–Schwarz inequality that

$$|f_x(z)| = |\langle z, x\rangle_H| \le \|z\|_H \|x\|_H \quad \text{for all } z \in H. \tag{1}$$

Now define the *Riesz operator* $R_H\colon H \to H'$ as $x \mapsto f_x$.

- R_H is linear: $R_H(ax_1 + bx_2) = f_{ax_1+bx_2} = \langle \cdot, ax_1 + bx_2\rangle_H = a\langle \cdot, x_1\rangle_H + b\langle \cdot, x_2\rangle_H = af_{x_1} + bf_{x_2} = aR_Hx_1 + bR_Hx_2$ for $x_1, x_2 \in H$, $a, b \in \mathbb{R}$.
- R_H is an isometry ($\|R_Hx\|_{H'} = \|x\|_H$): it follows from (1) that $\|R_Hx\|_{H'} = \|f_x\|_{H'} = \sup_{\|z\|_H \le 1} |\langle z, x\rangle_H| \le \|x\|_H$. The other direction follows from $\|x\|_H^2 = \langle x, x\rangle_H = f_x(x) = |f_x(x)| \le \|f_x\|_{H'}\|x\|_H = \|R_Hx\|_{H'}\|x\|_H$.
- R_H is injective (one-to-one): let $R_Hx = R_Hy$ for some $x, y \in H$. From linearity, $R_H(x-y) = 0 \Rightarrow f_{x-y} = 0 \Rightarrow \langle x-y, z\rangle_H = 0$ for all $z \in H \Rightarrow x = y$.
- R_H is surjective (onto): by Riesz representation theorem, given $A \in H'$, there exists a unique $x \in H$ satisfying $A(z) = \langle z, x\rangle_H = f_x(z)$ for all $z \in H$. In other words, $A = \langle \cdot, x\rangle_H = f_x = R_Hx$.

$\therefore$ The Riesz operator $R_H\colon H \to H'$ is a bijective linear operator isometry.

Lemma

Let H be a Hilbert space. The dual space $H' := \mathcal{L}(H, \mathbb{R})$ is a Hilbert space induced by

$$\|A\|_{H'} := \sup_{\|x\|_H \le 1} |Ax| = \sqrt{\langle A, A\rangle_{H'}}, \quad \langle A, B\rangle_{H'} := \langle R_H^{-1}A, R_H^{-1}B\rangle_H.$$

Adjoint operator

Proposition

Let H_1 and H_2 be real Hilbert spaces and suppose that $A \in \mathcal{L}(H_1, H_2)$. Then there exists a unique bounded linear operator $A^*: H_2 \to H_1$, called the adjoint *of A, satisfying $\langle Ax, y\rangle_{H_2} = \langle x, A^* y\rangle_{H_1}$. Moreover, $\|A\|_{H_1 \to H_2} = \|A^*\|_{H_2 \to H_1}$.*

Proof. Let $y \in H_2$ and consider $T_y : H_1 \to \mathbb{R}$, $x \mapsto \langle Ax, y\rangle_{H_2}$. Clearly, T_y is linear and bounded so by the Riesz representation theorem there exists a *unique* $z \in H_1$ s.t.

$$\langle Ax, y\rangle_{H_2} = T_y(x) = \langle x, z\rangle_{H_1} \quad \text{for all } x \in H_1.$$

Define $A^* y := z$.

- Let $a, b \in \mathbb{R}$ and $y_1, y_2 \in H_2$. Linearity follows from $\langle x, A^*(ay_1 + by_2)\rangle = \langle Ax, ay_1 + by_2\rangle = a\langle Ax, y_1\rangle + b\langle Ax, y_2\rangle = a\langle x, A^* y_1\rangle + b\langle x, A^* y_2\rangle = \langle x, aA^* y_1 + bA^* y_2\rangle$. Since $x \in H_1$ was arbitrary, $A^*(ay_1 + by_2) = aA^* y_1 + bA^* y_2$.
- $\|A^*\|_{H_2 \to H_1} = \sup_{\|y\|_{H_2} \le 1} \|A^* y\|_{H_1} \overset{(*)}{=} \sup_{\|y\|_{H_2} \le 1} \sup_{\|x\|_{H_1} \le 1} |\langle A^* y, x\rangle|$
  $= \sup_{\|y\|_{H_2} \le 1} \sup_{\|x\|_{H_1} \le 1} |\langle y, Ax\rangle| \overset{(*)}{=} \sup_{\|x\|_{H_1} \le 1} \|Ax\|_{H_2} = \|A\|_{H_1 \to H_2} < \infty$. □

(*) Let $\Lambda \in \mathcal{L}(H, K)$, H, K Hilbert spaces. Cauchy–Schwarz: $\sup_{\|y\|_K \le 1} |\langle \Lambda x, y\rangle_K| \le \|\Lambda x\|_K$.
Other direction: $\sup_{\|y\|_K \le 1} |\langle \Lambda x, y\rangle_K| \ge |\langle \Lambda x, \frac{1}{\|\Lambda x\|_K} \Lambda x\rangle|_K = \|\Lambda x\|_K$.
$\therefore \|\Lambda x\|_K = \sup_{\|y\|_K \le 1} |\langle \Lambda x, y\rangle_K|$.

Some properties of the adjoint operator

Proposition

Let H_1 and H_2 be real Hilbert spaces and suppose that $A, B \in \mathcal{L}(H_1, H_2)$. Then

(i) $\|A^*A\|_{H_1 \to H_1} = \|A\|^2_{H_1 \to H_2}$,

(ii) $A^{**} = A$, *where* $A^{**} = (A^*)^*$;

(iii) $(c_1 A + c_2 B)^* = c_1 A^* + c_2 B^*$, $c_1, c_2 \in \mathbb{R}$.

Proof. (i) Let $x \in H_1$, $\|x\|_{H_1} = 1$. By the Cauchy–Schwarz inequality,

$$\|Ax\|^2_{H_2} = \langle Ax, Ax\rangle_{H_2} = \langle x, A^*Ax\rangle_{H_1} \leq \|A^*Ax\|_{H_1} \;\Rightarrow\; \|A\|^2_{H_1\to H_2} \leq \|A^*A\|_{H_1 \to H_1}.$$

Other direction: $\|A^*A\| \leq \|A^*\| \cdot \|A\| = \|A\|^2$ (previous slide and exercise of week 1).
(ii) If $x \in H_1$ and $y \in H_2$, then

$$\langle A^{**}x, y\rangle_{H_2} = \langle x, A^*y\rangle_{H_1} = \langle A^*y, x\rangle_{H_1} = \langle y, Ax\rangle_{H_2} = \langle Ax, y\rangle_{H_2}.$$

Hence $\langle A^{**}x - Ax, y\rangle_{H_2} = 0$ for all $y \in H_2 \Rightarrow A^{**}x = Ax$ for all $x \in H_1 \Rightarrow A^{**} = A$.
(iii) Let $x \in H_1$ and $y \in H_2$. Then

$$\begin{aligned}\langle (c_1A + c_2B)^*y, x\rangle_{H_1} &= \langle y, (c_1A + c_2B)x\rangle_{H_2} = c_1\langle y, Ax\rangle_{H_2} + c_2\langle y, Bx\rangle_{H_2} \\ &= c_1\langle A^*y, x\rangle_{H_1} + c_2\langle B^*y, x\rangle_{H_1} = \langle (c_1A^* + c_2B^*)y, x\rangle_{H_1}.\end{aligned}$$

Similarly to the previous part, we conclude that $(c_1A + c_2B)^* = c_1A^* + c_2B^*$. □

Self-adjoint operators

Definition

Let H be a Hilbert space. The operator $A \in \mathcal{L}(H) := \mathcal{L}(H, H)$ is called *self-adjoint* if $A^* = A$, i.e.,

$$\langle Ax, y \rangle = \langle x, Ay \rangle \quad \text{for all } x, y \in H.$$

Example

Let H be a Hilbert space and let $A, B \in \mathcal{L}(H)$ be self-adjoint operators. Then

(i) $A + B$ is self-adjoint.

(ii) if $c \in \mathbb{R}$, then cA is self-adjoint.

(iii) if $AB = BA$, then AB is self-adjoint.

Parts (i) and (ii) follow immediately from part (iii) on the previous slide. If $x, y \in H$, then

$$\langle ABx, y \rangle = \langle BAx, y \rangle = \langle Ax, By \rangle = \langle x, ABy \rangle \quad \Rightarrow \quad (AB)^* = AB.$$

Example

Let H be a Hilbert space and $M \subset H$ a closed subspace. Then the orthogonal projections $P_M \colon H \to M$ and $I - P_M =: P_{M^\perp} : H \to M^\perp$ are self-adjoint.

Lax–Milgram lemma

Proposition (Lax–Milgram lemma)

Let H be a real Hilbert space and let $B\colon H \times H \to \mathbb{R}$ be a bilinear mapping[†] *with $C, c > 0$ such that*

$$|B(u,v)| \leq C\|u\| \cdot \|v\| \quad \textit{for all } u, v \in H, \qquad \text{(boundedness)}$$

$$B(u,u) \geq c\|u\|^2 \quad \textit{for all } u \in H. \qquad \text{(coercivity)}$$

Let $F\colon H \to \mathbb{R}$ be a bounded linear mapping. Then there exists a unique element $u \in H$ satisfying

$$B(u,v) = F(v) \quad \textit{for all } v \in H.$$

and

$$\|u\| \leq \frac{1}{c}\|F\|.$$

[†] $B(u+v,w) = B(u,w) + B(v,w),\ B(au,v) = aB(u,v),$
$B(u,v+w) = B(u,v) + B(u,w),\ B(u,av) = aB(u,v)$
for all $u, v, w \in H$ and $a \in \mathbb{R}$.

Proof. 1) Let $v \in H$ be fixed. Then the mapping

$$T\colon w \mapsto B(v, w),\ H \to \mathbb{R},$$

is bounded and linear. It follows from the Riesz representation theorem that there exists a unique element $a \in H$ with

$$Tw = \langle a, w \rangle \quad \text{for all } w \in H.$$

Let us define the mapping $A\colon H \to H$ by setting

$$Av = a.$$

Then

$$B(v, w) = \langle Av, w \rangle \quad \text{for all } v, w \in H.$$

2) We show that the mapping $A\colon H \to H$ is linear and bounded. Clearly,

$$\begin{aligned}\langle A(c_1 v_1 + c_2 v_2), w\rangle &= B(c_1 v_1 + c_2 v_2, w) \\ &= c_1 B(v_1, w) + c_2 B(v_2, w) \\ &= \langle c_1 A v_1 + c_2 A v_2, w\rangle\end{aligned}$$

for all $w \in H$, so $A(c_1 v_1 + c_2 v_2) = c_1 A v_1 + c_2 A v_2$. Moreover,

$$\begin{aligned}\|Av\|^2 &= \langle Av, Av\rangle \\ &= B(v, Av) \\ &\leq C\|v\|\|Av\|\end{aligned}$$

which implies that

$$\|Av\| \leq C\|v\|.$$

3) We show that

$$\begin{cases} A \text{ is one-to-one,} \\ \mathrm{Ran}(A) = AH \text{ is closed in } H. \end{cases}$$

We begin by noting that

$$c\|v\|^2 \leq B(v, v) = \langle Av, v\rangle \leq \|Av\|\|v\|$$

and thus

$$\|Av\| \geq c\|v\| \quad \text{for all } v \in H. \tag{2}$$

Especially

$$Av = Aw \Rightarrow A(v - w) = 0 \Rightarrow 0 = \|A(v - w)\| \geq c\|v - w\| \geq 0 \Rightarrow v = w$$

so A is one-to-one.

To see that Ran(A) is closed, let $y_j = Ax_j \in \mathrm{Ran}(A)$. The goal is to show that $y := \lim_{j\to\infty} y_j \in \mathrm{Ran}(A)$. We observe that

$$\lim_{j,k\to\infty} \|x_j - x_k\| \overset{(2)}{\leq} \lim_{j,k\to\infty} \frac{1}{c}\|y_j - y_k\| = 0,$$

i.e., $(x_j)_{j=1}^{\infty}$ is Cauchy and $x := \lim_{j\to\infty} x_j \in H$ exists by completeness. Moreover,

$$\lim_{j\to\infty} \|Ax_j - Ax\| \leq \lim_{j\to\infty} \|A\|\|x_j - x\| \leq C \lim_{j\to\infty} \|x_j - x\| = 0$$

and therefore

$$y = \lim_{j\to\infty} Ax_j = Ax \in \mathrm{Ran}(A).$$

4) We show that $\mathrm{Ran}(A) = H$. We prove this by contradiction: suppose that $\mathrm{Ran}(A) = \overline{\mathrm{Ran}(A)} \neq H$. Then there exists $w \in \mathrm{Ran}(A)^\perp$, $w \neq 0$.[†] This implies that

$$\|w\|^2 \leq \frac{1}{c} B(w, w) = \frac{1}{c} \langle Aw, w \rangle = 0,$$

i.e., $w = 0$. This contradiction shows that $\mathrm{Ran}(A) = H$. Therefore $A \colon H \to H$ is a continuous bijection.

5) Existence of a solution. We use the Riesz representation theorem: since $F \colon H \to \mathbb{R}$ is linear and continuous, there exists $b \in H$ such that

$$F(v) = \langle b, v \rangle \quad \text{for all } v \in H.$$

Define $u := A^{-1} b$. Hence

$$\begin{aligned} Au = b \quad &\Leftrightarrow \quad \langle Au, v \rangle = \langle b, v \rangle \quad \text{for all } v \in H \\ &\Leftrightarrow \quad B(u, v) = F(v) \quad \text{for all } v \in H. \end{aligned}$$

[†] Since $(\mathrm{Ran}(A)^\perp)^\perp = \overline{\mathrm{Ran}(A)} \neq H \Rightarrow (\mathrm{Ran}(A))^\perp \neq \{0\}$.

6) Uniqueness. Suppose that

$$B(u_1, w) = F(w) \quad \text{for all } w \in H,$$
$$B(u_2, w) = F(w) \quad \text{for all } w \in H.$$

Let $u := u_1 - u_2$. By linearity,

$$B(u, w) = 0 \quad \text{for all } w \in H.$$

The coercivity of B implies that

$$\|u\|^2 \leq \frac{1}{c} B(u, u) = 0$$

so that $u = 0$, i.e., $u_1 = u_2$.

7) *A priori bound*. If $B(u, w) = F(w)$ for all $w \in H$, then by setting $w = u$ we obtain

$$\|u\|^2 \leq \frac{1}{c} B(u, u) = \frac{1}{c} F(u) \leq \frac{1}{c} \|F\| \|u\|$$

which immediately yields

$$\|u\| \leq \frac{1}{c} \|F\|.$$

□

Density argument

Lemma

Let X, Y be Banach spaces and let $Z \subset X$ be a dense subspace. If $T: Z \to Y$ is a linear mapping such that

$$\|Tx\|_Y \leq C\|x\|_X, \quad x \in Z, \tag{3}$$

then there exists a unique extension $\widetilde{T}: X \to Y$ with $\widetilde{T}|_Z = T$ and

$$\|\widetilde{T}x\|_Y \leq C\|x\|_X, \quad x \in X. \tag{4}$$

Moreover, if (3) holds with equality, then so does (4).

Proof. Let $x \in X$. Because $Z \subset X$ is dense, there exists a sequence $(z_k)_{k=1}^{\infty} \subset Z$ s.t. $\|z_k - x\|_X \xrightarrow{k\to\infty} 0$. Let $\varepsilon > 0$. Since $(z_k)_{k=1}^{\infty}$ is a Cauchy sequence, there exists $N \in \mathbb{N}$ s.t.

$$m, n \geq N \quad \Rightarrow \quad \|z_m - z_n\|_X < \frac{\varepsilon}{C}.$$

Then there holds

$$\|Tz_m - Tz_n\|_Y = \|T(z_m - z_n)\|_Y \leq C\|z_m - z_n\|_X < \varepsilon,$$

which means that $(Tz_k)_{k=1}^{\infty}$ is a Cauchy sequence in Y. Since Y is complete, there exists $y := \lim_{k\to\infty} Tz_k$. Hence we may define $\widetilde{T}: X \to Y$ by setting $\widetilde{T}(x) = y$.

We begin by showing that $\widetilde{T}$ is well-defined. Let $(z_k)_{k=1}^{\infty}$, $(\widetilde{z}_k)_{k=1}^{\infty}$ be two sequences in Z s.t. $z_k, \widetilde{z}_k \xrightarrow{k\to\infty} x$ in X. Then

$$\|Tz_k - T\widetilde{z}_k\|_Y = \|T(z_k - \widetilde{z}_k)\|_Y \leq C\|z_k - \widetilde{z}_k\| \leq C\|z_k - x\| + C\|\widetilde{z}_k - x\| \overset{k\to\infty}{\to} 0.$$

Recalling that $\widetilde{T}(x) := \lim_{k\to\infty} Tz_k$, we obtain

$$\|T\widetilde{z}_k - \widetilde{T}(x)\| \leq \|T\widetilde{z}_k - Tz_k\| + \|Tz_k - \widetilde{T}(x)\| \overset{k\to\infty}{\to} 0,$$

showing that $\widetilde{T}$ is well-defined.
Next we show that $\widetilde{T}$ is linear. Let $x, \widetilde{x} \in X$ and $a, b \in \mathbb{R}$. Let $Z \ni z_k \xrightarrow{k\to\infty} x$ and $Z \ni \widetilde{z}_k \xrightarrow{k\to\infty} \widetilde{x}$. Now $ax + b\widetilde{x} \in X$ and $Z \ni az_k + b\widetilde{z}_k \to ax + b\widetilde{x}$. Thus

$$\widetilde{T}(ax + b\widetilde{x}) = \lim_{k\to\infty} T(az_k + b\widetilde{z}_k) = a \lim_{k\to\infty} Tz_k + b \lim_{k\to\infty} T\widetilde{z}_k = a\widetilde{T}x + b\widetilde{T}x,$$

since the limit is linear.[†]
Since the norm is continuous,

$$\|\widetilde{T}x\| = \|\lim_{k\to\infty} Tx_k\| = \lim_{k\to\infty} \|Tx_k\| \leq C \lim_{k\to\infty} \|x_k\| = C\|x\|.$$

Finally, $\widetilde{T}|_Z = T$ holds by construction and the uniqueness of the limit $Tz_k \to y$ ensures that there cannot exist another mapping $L\colon X \to Y$ s.t. $L|_Z = T$ and $\|Lx\| \leq C\|x\|$. □

[†]Let $y := \lim_{k\to\infty} Tz_k$ and $\widetilde{y} := \lim_{k\to\infty} T\widetilde{z}_k$.
Then $\|T(az_k + b\widetilde{z}_k) - ay - b\widetilde{y}\| \leq a\|Tz_k - y\| + b\|T\widetilde{z}_k - \widetilde{y}\| \to 0$.
Hence $\lim_{k\to\infty} T(az_k + b\widetilde{z}_k) = a\lim_{k\to\infty} Tz_k + b\lim_{k\to\infty} T\widetilde{z}_k$.